A Leslie-type urban-rural migration model, and the situation of Germany and Turkey

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Abstract Movements in the age structure of a population are often accompanied by substantial rural-urban migration. It is therefore compelling to analyze the implications of fertility, mortality, and migration patterns together. We use a joint Leslie-type population model of urban and rural populations which projects the current population structure into the future, allowing for migration in both directions. This model permits an analysis of the long-run (stable) population properties, such as urbanization, under the assumption that current conditions persist, as well as an analysis of rural and urban populations in isolation, when migration is computationally eliminated. Applying the model to the female populations below age 50 of Germany and Turkey, we find that the actual urbanization is lower (higher) than long-run urbanization in Germany (Turkey, respectively). Among our findings is also that the slight long-run growth of the Turkish population is due to rural-urban migration, while Turkish urban areas have a below-replacement fertility.

Keywords Leslie-type model · Population projection · Urban-rural migration · Stable population theory · Population growth · Urbanization · Germany · Turkey

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1 Introduction

"The world is undergoing the largest wave of urban growth in history", the United Nations Population Fund warns in an 2007 online release.[1] Meanwhile, by the year 2010, world urbanization has arrived at the 50% level, with a five-year urban population growth of 1.9% (compared to an overall population growth of 1.2%), which is an aggregate of values spanning from 0.7 for more developed regions to 4.0 for least-developed countries, according to the UNFPA State of World Population report 2010 [23].

Urbanization and economic growth are often closely linked, but urbanization as well concentrates poverty. Urbanization results from internal migration. Therefore, on the other hand, the increasing urbanization is insofar problematic as it contributes to the aggravation of the structural weakness of rural regions which may just be a major push factor of rural-urban migration. Urbanization may also be linked to decreasing fertility and shrinking, hence ageing populations, as there is a gap in fertility levels between urban and rural regions throughout the world.[2] The present study compares Germany ("developed" with respect to urbanization as well as growth, according to the UNFPA report [23]) and Turkey (also "developed" with respect to urbanization, but "less developed" with respect to growth, according to the UNFPA report [23]).

Germany, among the more developed countries, is attributed a 74% level of urbanization in the UNFPA report [23], along with an indiscernible urban growth (the reported growth rate is 0.0%) within five years. Accelerated urbanization in Germany became apparent from the mid-nineteenth century onward; it had its origin in industrialization and its impulse to the formation of new urban settlements (which, according to Köllmann [10], became possible only with the abolition of the older municipal constitutions of "guild and trade restrictions designed to discourage migration"). Tracing the trends of internal migration in German history, Mai et al. [15] distinguish a phase of long-distance migration (from rural areas in the East to urban areas in the West), progressively replacing migration from urban hinterland, so that by the early twentieth century about 50% of the German population had become internal migrants, as Köllmann [11] calculates. Between the wars, for reasons of supply and ideology, beginnings of a trend back to the countryside could be observed. When the World War II aftermath with its reallocation of population had abated in the late 1950s, a phase of suburbanization solidified in Western Germany, leading to the emergence of urban sprawls and a persisting large-scale de-concentration of population. However, rural-urban migration remained relevant particularly for the young. — The situation in East Germany before 1990 was different, where regional concentration of population had prevailed. A sharp drop in younger age groups could be witnessed in the 1990s, which was triggered in particular by east-west migration after German

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[1] http://www.unfpa.org/pds/urbanization.htm, accessed: 2011-05-14

[2] l.c.

reunification and amplified by low fertility rates; cf. [5], [15]. Thus, suburbanization trends in East Germany have attenuated by now, with urban sprawls gradually coming into being.

The ongoing phenomenon of rural depopulation used to hit east-German regions in particular, but not exclusively. It is selective with respect to the younger age of migrants, and is accompanied by low total fertility levels (1.33 in 2010; cf. [23]) having persisted for more than three decades. Mai et al. [15] assert that total growth of the German population today rests solely on an increase in life expectancy, the ageing of the population being notably pronounced in rural areas.

With a 70% proportion of population living in urban areas today, Turkey is on a level with more developed regions like Germany, while its five-year urban growth rate of 1.9% is characteristic for less-developed countries according to the 2010 UNFPA report [23]. With delay from the trend in Germany, urbanization in Turkey started on a large scale in the early 1950s from a level of 20% only. For the Turkish population between 1955 and 2000, Gedik [6] investigated the Alonso theory of differential urbanization which postulates cycles of three evolutionary phases, urbanization, polarization reversal, and counter urbanization, where growth rates are highest for large, medium, and small settlement sizes, respectively. Gedik found evidence for a phase of pre-concentration in small cities in the 1950s, large-city urbanization that followed, and polarization "dispersal" starting in 1980 with highest growth rates in medium-sized cities dispersed throughout the country.

Simultaneously, Turkey experienced a pronounced change in fertility. Within three decades, the total fertility rate halved down to barely 2 today (2.09 in 2010, c.f. [23]). The Turkey Demographic and Health Survey 2008 [8] investigates fertility preferences and behavior of Turkish women by residence, and provides information on maternal and child health. Among the findings is that, though total urban fertility is below replacement and the urban-rural differential appears to be contracting over time, clear above-replacement levels in South and East Anatolia persist. At each age class, rural women tend to bear more children than women in urban centers where a trend that fertility decreases with a higher educational level can be observed. As a result, the urban-rural gap with respect to the median age at first birth is broadening. A significantly higher proportion in urban centers of women in the working ages certainly adds to this gap, and thus the effects of rural-to-urban migration of economically active women. Furthermore, the findings of the survey indicate a significant urban-rural differential in child mortality which appears to be correlated with the mother's young age and educational level.

From a formal point of view, the phenomenon of urbanization (respectively rural depopulation) can be analyzed as a result of migration probabilities and their interaction with urban- and rural-specific fertility rates and survival probabilities. The approach may vary in several basic respects: The focus may be either on forecasting the future population using forecasts of mortality, fertility and migration, or on population projection in order to find answers to

what the population would be like in the long run if mortality, fertility and migration would evolve (or persist) in a certain way. The effects of demographic and environmental conditions on the dynamics of populations may be studied in discrete or continuous time, in a deterministic framework, or using a probabilistic model with random variation in births, deaths, and migration.

In her overview of probabilistic approaches to demographic and population forecasting, Booth [2] identifies three widely-used frameworks: methods on the basis of sample data on individual expectations about future developments or expert opinions, structural modeling methods based on theories on relations between demographic variables and processes, and extrapolative methods using time series models to detect patterns and trends in the past and extrapolate them into the future. A time series approach recently adopted by Hyndman, e.g. [9], involves functional data models to forecast mortality, fertility, and migration.

The Leslie population model (in recognition of Leslie's work, cf. [14]) falls under the category of projection. In its classical formulation, it is a discrete-time and age-structured (respectively stage-structured, cf. [13]) transition matrix model for the evolution of a closed population in time, but can be extended to a Leslie-type model which allows for immigration; e.g. [4], [20]. The spectral properties of the matrix provide insight into asymptotic population growth rates and stable stage structures. A detailed review of matrix population models, including Leslie models for populations in time-varying, deterministic or stochastic environments, is given by Caswell [3]. Many applications and developments in spatial demography have their root in the (also matrix-based) multiregional cohort-component model introduced by Rogers (cf. [18], [19]), which, accessing a multispatial life table, describes the dynamics of a population which is dispersed over different spatial patches and allows for migration in between. An example is the multiregional multinational cohort-component projection model by Kupiszewska and Kupiszewski, cf. [12] for its revised form to capture international migration. For an ecological system in a multi-patch stochastic environment with different time scales for migration and vital rates, recently Alonso and Sanz [1] showed the application of an aggregation method in order to obtain a reduced stochastic Leslie model.

Our contribution is a mathematical model conceived for urban and rural populations, which is an extension of the classical Leslie model and allows for migration from rural to urban areas and in the opposite direction. It can be understood as a particular formulation of the multiregional model by Rogers [19] installing two regional types, urban and rural, but at the same time two kinds of inhabitants, natives and migrants. It is able to project the current urbanization structure into the future and permits sensitivity analyses of the impact of different vital patterns and migration scenarios on population growth and urbanization, as well as insight into the trade-off between fertility, mortality and migration with respect to stability.

This model is introduced in Section 2. First, a hypothetical population is considered in Section 3, then an application to the populations of Germany and Turkey is presented in Section 4. Finally, Section 5 gives a summary and

some outline for further research. A further challenge in our studies was how we may obtain the information needed as input to our model. The data used in our study, including data sources and data processing, are specified in the Appendix. All computations are carried out in R [17].

2 The model

Our model deals with four populations of females: city natives, village natives, and two classes of migrants which are distinguished by destination into city migrants and village migrants. Time proceeds in discrete steps; for illustration purposes we choose 15 years in this and the following section (but 5-year steps when applied to Germany and Turkey in Section 4). Accordingly, the four populations are structured by three 15-year intervals of age covering reproductive ages. The age-specific fertility, mortality and migration patterns are assumed to be constant through time. They may be village- or city-specific. In particular, the migrants' vital rates may differ from those of the natives as well, while second generation migrants are assumed to behave like natives with this respect; they are counted as natives actually.

Our model is defined by the relationship and the matrix displayed in Table 1; all symbols are defined in Table 2. Schematically, the matrix M_I can also be written as

$$
\begin{pmatrix}
M_I{}^{[\text{city native}]}_{[\text{city native}]} & M_I{}^{[\text{city migrant}]}_{[\text{city native}]} & M_I{}^{[\text{village native}]}_{[\text{city native}]} & M_I{}^{[\text{village migrant}]}_{[\text{city native}]} \\
M_I{}^{[\text{city native}]}_{[\text{city migrant}]} & M_I{}^{[\text{city migrant}]}_{[\text{city migrant}]} & M_I{}^{[\text{village native}]}_{[\text{city migrant}]} & M_I{}^{[\text{village migrant}]}_{[\text{city migrant}]} \\
M_I{}^{[\text{city native}]}_{[\text{village native}]} & M_I{}^{[\text{city migrant}]}_{[\text{village native}]} & M_I{}^{[\text{village native}]}_{[\text{village native}]} & M_I{}^{[\text{village migrant}]}_{[\text{village native}]} \\
M_I{}^{[\text{city native}]}_{[\text{village migrant}]} & M_I{}^{[\text{city migrant}]}_{[\text{village migrant}]} & M_I{}^{[\text{village native}]}_{[\text{village migrant}]} & M_I{}^{[\text{village migrant}]}_{[\text{village migrant}]}
\end{pmatrix}
\tag{1}
$$

with sub-matrices $M_I{}^{[\text{from}]}_{[\text{to}]}$ indicating possible transitions within one time step. Such a sub-matrix will be a zero matrix whenever the corresponding transition is impossible (as in the case $M_I{}^{[\text{city native}]}_{[\text{city migrant}]}$). The element-by-element sum of submatrices along a column of the partitioned M_I will result in a usual Leslie matrix, which in turn defines population streams channeled to several possible destinations by means of the migration pattern.

Theorem: Consider an age-structured population evolving according to $N_t = M_I \cdot N_{t-1}$.
Let the following conditions be satisfied: (i) migration probabilities from city to village and from village to city are positive for any (not necessarily the same) age class, (ii) survival probabilities are all positive, and (iii) fertility rates are positive for any two adjacent age classes. Then:

a) There exists a stable population structure $\tilde{N}$ and a $\lambda \in \mathbb{R}$ such that

$$\lambda \cdot \tilde{N} = M_I \cdot \tilde{N}. \tag{2}$$

Here, λ is the maximum eigenvalue of M_I.

b) The future long-run growth rate of an initial population is given by the maximum eigenvalue of M_I.

c) All four population segments (city native, city migrant, village native, village migrant) will ultimately grow with the same rate.

Proof: The projection matrix M_I is a non-negative square matrix, irreducible and primitive. Therefore, classical Perron-Frobenius theory can be applied, see e.g. Seneta [21] and Caswell [3].

Irreducibility and primitivity can be evaluated from the transition diagrams in Figures 1 and 2: The "life cycle graph" is strongly connected, i.e. each pair of nodes is connected in the sense that one node can be reached from the other within a finite number of transitions. The greatest common divisor of loop lengths is 1. This holds if only the conditions of the theorem are satisfied.

Then, according to the Perron-Frobenius theorem, there exists a real and positive eigenvalue λ which dominates any other eigenvalue of M_I. The eigenvectors associated to λ are strictly positive and unique to constant multiples. In particular, there exists a right eigenvector $\tilde{N}$ such that equation (2) holds. It follows what is known as the strong ergodic theorem, that λ completely determines the long-term dynamics of the population:

$$\lim_{t\to\infty} \frac{N_t}{\lambda^t} = c \cdot \tilde{N} \tag{3}$$

In the long run, the population will grow at a rate given by λ, with a stable population structure proportional to $\tilde{N}$. In particular, this ultimate growth rate carries over to all four population segments. $\square$

$$
\underbrace{\begin{pmatrix}
n_{c1t} \\ n_{c2t} \\ n_{c3t} \\ \hline n^*_{c1t} \\ n^*_{c2t} \\ n^*_{c3t} \\ \hline n_{v1t} \\ n_{v2t} \\ n_{v3t} \\ \hline n^*_{v1t} \\ n^*_{v2t} \\ n^*_{v3t}
\end{pmatrix}}_{\substack{\text{city native}\\[2pt]\text{city migrant}\\[2pt]\text{village native}\\[2pt]\text{village migrant}}}
=
\left(\begin{array}{ccc|ccc|ccc|ccc}
f_{c1}\bar{m}_{c1} & f_{c2}\bar{m}_{c2} & f_{c3}\bar{m}_{c3} & f^*_{c1}\bar{m}^*_{c1} & f^*_{c2}\bar{m}^*_{c2} & f^*_{c3}\bar{m}^*_{c3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{c1}\bar{m}_{c1} & 0 & 0 & p^*_{c1}\bar{m}^*_{c1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{c2}\bar{m}_{c2} & 0 & 0 & p^*_{c2}\bar{m}^*_{c2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & f_{v1}m_{v1} & f_{v2}m_{v2} & f_{v3}m_{v3} & f^*_{v1}m^*_{v1} & f^*_{v2}m^*_{v2} & f^*_{v3}m^*_{v3} \\
0 & 0 & 0 & 0 & 0 & 0 & p_{v1}m_{v1} & 0 & 0 & p^*_{v1}m^*_{v1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{v2}m_{v2} & 0 & 0 & p^*_{v2}m^*_{v2} & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & f_{v1}\bar{m}_{v1} & f_{v2}\bar{m}_{v2} & f_{v3}\bar{m}_{v3} & f^*_{v1}\bar{m}^*_{v1} & f^*_{v2}\bar{m}^*_{v2} & f^*_{v3}\bar{m}^*_{v3} \\
0 & 0 & 0 & 0 & 0 & 0 & p_{v1}\bar{m}_{v1} & 0 & 0 & p^*_{v1}\bar{m}^*_{v1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_{v2}\bar{m}_{v2} & 0 & 0 & p^*_{v2}\bar{m}^*_{v2} & 0 \\ \hline
f_{c1}m_{c1} & f_{c2}m_{c2} & f_{c3}m_{c3} & f^*_{c1}m^*_{c1} & f^*_{c2}m^*_{c2} & f^*_{c3}m^*_{c3} & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{c1}m_{c1} & 0 & 0 & p^*_{c1}m^*_{c1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{c2}m_{c2} & 0 & 0 & p^*_{c2}m^*_{c2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\cdot
\begin{pmatrix}
n_{c1,t-1} \\ n_{c2,t-1} \\ n_{c3,t-1} \\ \hline n^*_{c1,t-1} \\ n^*_{c2,t-1} \\ n^*_{c3,t-1} \\ \hline n_{v1,t-1} \\ n_{v2,t-1} \\ n_{v3,t-1} \\ \hline n^*_{v1,t-1} \\ n^*_{v2,t-1} \\ n^*_{v3,t-1}
\end{pmatrix}
$$

Table 1 The relation $N_t = M_I \cdot N_{t-1}$

Symbol	Definition
f_{ci}	the average number of girls born to a native of city in age class i, and surviving to the next age class
f_{vi}	the average number of girls born to a native of village in age class i, and surviving to the next age class
f_{ci}^*	the average number of girls born to a migrant (from village to city) in age class i, and surviving to the next age class
f_{vi}^*	the average number of girls born to a migrant (from city to village) in age class i, and surviving to the next age class
p_{ci}	the probability that a native of city now in age class i, survives to be in $i+1$
p_{vi}	the probability that a native of village now in age class i, survives to be in $i+1$
p_{ci}^*	the probability that a migrant (from village to city) now in age class i survives to be in $i+1$
p_{vi}^*	the probability that a migrant (from city to village) now in age class i survives to be in $i+1$
m_{ci}	probability that a city native in age group i migrates to village
$\bar{m}_{ci}$	probability that a city native in age group i does not migrate to village; $\bar{m}_{ci} = 1 - m_{ci}$
m_{vi}	probability that a village native in age group i migrates to city
$\bar{m}_{vi}$	probability that a village native in age group i does not migrate to city; $\bar{m}_{vi} = 1 - m_{vi}$
m_{ci}^*	probability that a migrant (from village to city) in age group i migrates again (back to village)
$\bar{m}_{ci}^*$	probability that a migrant (from village to city) in age group i does not migrate again (back to village); $\bar{m}_{ci}^* = 1 - m_{ci}^*$
m_{vi}^*	probability that a migrant (from city to village) in age group i migrates again (back to city)
$\bar{m}_{vi}^*$	probability that a migrant (from city to village) in age group i does not migrate again (back to city); $\bar{m}_{vi}^* = 1 - m_{vi}^*$
n_{ct}	age structured city population of natives in period t
n_{vt}	age structured village population of natives in period t
n_{ct}^*	age structured population of migrants from village to city in period t
n_{vt}^*	age structured population of migrants from city to village in period t

Table 2 Explanation of symbols in the model

3 A hypothetical example

The following hypothetical example is meant to illustrate the dynamics of the model outlined in Section 2, and to demonstrate how the model can contribute to analyzing the successive development of a population. The example is based on a population broken down into three 15-year age classes: 0–15, 15–30 and
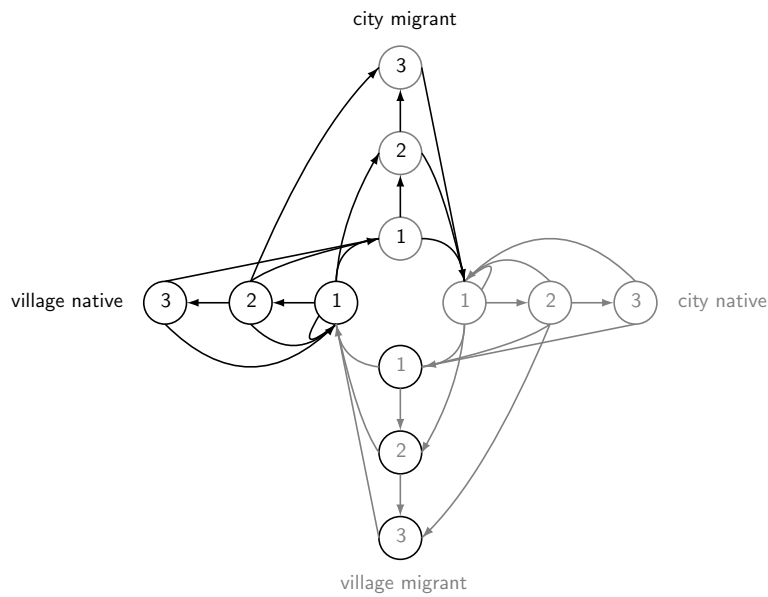
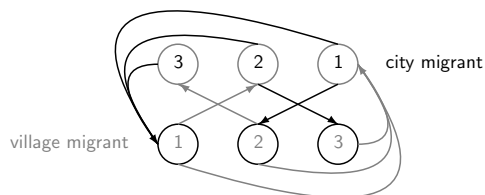**Fig. 1** Transitions between states (except migrant to migrant transitions)

Fig. 2 Transitions between migrant states

30–45. The first step in constructing the population dynamics is to define two Leslie matrices describing two populations, "city" and "village", in isolation (with no migration between them). The two populations are then linked together by specifying fertility rates and survival probabilities for migrants and, most crucially for our purposes, a migration pattern between them. In what follows, we shall investigate two among many possible migration patterns with respect to the stable development of the resulting population. When analyzing actual populations, the migration pattern has to be inferred from population statistics, and in this case the procedure of linking populations together can be reversed to study city and village populations in isolation, as we shall see in Section 4.

3.1 Two populations in isolation

Let two Leslie matrices be given as

$$M_{\text{city}} = \begin{pmatrix} 0.10 & 0.20 & 0.20 \\ 0.95 & 0 & 0 \\ 0 & 0.90 & 0 \end{pmatrix}, \quad M_{\text{village}} = \begin{pmatrix} 0.30 & 0.90 & 0.70 \\ 0.90 & 0 & 0 \\ 0 & 0.85 & 0 \end{pmatrix}. \quad (4)$$

Their respective maximum eigenvalues are: $\lambda_{\text{city}} = 0.7086$, $\lambda_{\text{village}} = 1.2699$. The resulting growth of the stable populations in a 15-year interval is therefore -29.14% (city) and $+26.99\%$ (village), corresponding to annual rates of -2.27% and $+1.61\%$, respectively. When considered in isolation, the city population is thus shrinking, while the village population is growing (when stability is reached). The next step is to link the two populations together via specification of a migration pattern between them. The two cases we consider are: (i) there is no further migration after migrating once, (ii) migration is location-specific.

3.2 Linkage I: no migration back to the origin

The assumptions connecting city and village populations are:

– **Assumption 1:** Migration probabilities may depend on the origin (city or village), but are equal across age classes:

$$m_{c1} = m_{c2} = m_{c3} = m_c, \quad m_{v1} = m_{v2} = m_{v3} = m_v. \quad (5)$$

– **Assumption 2:** A migrant will stay at her destination and won't migrate back:

$$m_{c1}^* = m_{c2}^* = m_{c3}^* = 0, \quad m_{v1}^* = m_{v2}^* = m_{v3}^* = 0. \quad (6)$$

– **Assumption 3:** Migrants' survival probabilities are obtained as arithmetic means of city native and village native survival probabilities in their respective age class:

$$p_{ci}^* = p_{vi}^* = 0.5 \cdot (p_{ci} + p_{vi}), \quad i = 1, 2, 3. \quad (7)$$

– **Assumption 4:** As far as fertility is concerned, migrants will "split the difference" between city and village fertility, that is, the age-specific fertility rate of migrants is obtained as the arithmetic mean of the respective age-specific fertility rates of city and village population. In symbols:

$$f_{ci}^* = f_{vi}^* = 0.5 \cdot (f_{ci} + f_{vi}), \quad i = 1, 2, 3. \quad (8)$$

This leads to the matrix M_I displayed in Table 3. A comparison of the matrices M_I in Tables 1 and 3 reveals that $M_I{}_{[\text{village migrant}]}^{[\text{city migrant}]}$ and $M_I{}_{[\text{city migrant}]}^{[\text{village migrant}]}$ are now zero matrices, reflecting our assumption that migration back is impossible ($m_c^* = m_v^* = 0$). Considering the growth properties of city (shrinking) and village (growing) populations, the theorem in Section 2 implies that there is

$$\begin{pmatrix}
0.10\bar{m}_c & 0.20\bar{m}_c & 0.20\bar{m}_c & 0.20 & 0.55 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.95\bar{m}_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.90\bar{m}_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.30m_v & 0.90m_v & 0.70m_v & 0 & 0 & 0 \\
0 & 0 & 0 & 0.90 & 0 & 0 & 0.90m_v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.85 & 0 & 0 & 0.85m_v & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.30\bar{m}_v & 0.90\bar{m}_v & 0.70\bar{m}_v & 0.20 & 0.55 & 0.45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.90\bar{m}_v & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.85\bar{m}_v & 0 & 0 & 0 & 0 \\
0.10m_c & 0.20m_c & 0.20m_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.95m_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 \\
0 & 0.90m_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.90 & 0
\end{pmatrix}$$

Table 3 Matrix M_I, resulting from the assumptions in the example

a migration pattern, expressed by m_c and m_v, that will ultimately lead to a *stationary* total population. The transient behavior of the population is illustrated in Figure 3, with plots of the 12 series (four population groups: native city, migrant city, native village, migrant village; each broken down into three age groups) in $N_t = M_I \cdot N_{t-1}$ for $t = 1, \dots, 25$, where the initial population is given by

$$N_0' = (\underbrace{1000, 1000, 1000,}_{\text{city native}} \underbrace{0, 0, 0}_{\text{city migrant}}, \underbrace{1000, 1000, 1000,}_{\text{village native}} \underbrace{0, 0, 0}_{\text{village migrant}}), \qquad (9)$$

and with migration probabilities $m_c = 0.1$ (from city to village) and $m_v = 0.3$ (from village to city) The maximum eigenvalue of M_I, as displayed in Table 3, equals $\lambda = 0.9920$, so that the population shrinks in the long run: The low fertility of the city population prevails due to the high village-to-city migration ($m_v = 0.3$). Lowering m_v somewhat (to $m_v = 0.28$, say) would make the population grow.

In which way do growth and urbanization of the population in the long run depend on fertility? This question can be discussed by letting stable growth and urbanization depend on two factors, designated as f_c factor and f_v factor in Figure 4, with which fertility rates f_{ci}, f_{ci}^* and f_{vi}, f_{vi}^*, respectively, are multiplied. This method of investigating the impact of fertility is suitable because fertility rates have no theoretical upper bound. Figure 4 shows the contour lines of growth (that is, the maximum eigenvalue of M_I, hence the growth in a 15-year interval) and urbanization when city-specific and village-specific fertility rates are varied in a range from -50% to $+100\%$ of their original levels (Table 3).

Urbanization is computed as the share of population in the first six components of the population vector. (Technically, urbanization is the sum of the first six components of the right eigenvector belonging to the maximum eigenvalue, divided by the total sum.) The higher slope of the growth surface along the f_v factor axis points to the more important role of *village* fertility for growth under the given mortality and migration regime. A higher village fertility can offset a higher city fertility with respect to urbanization, at the same time leading to higher growth.
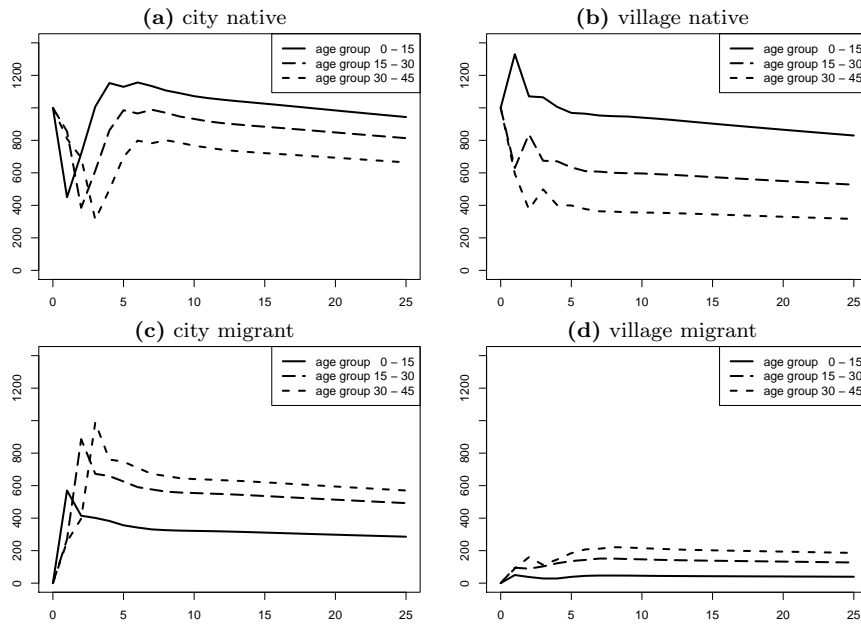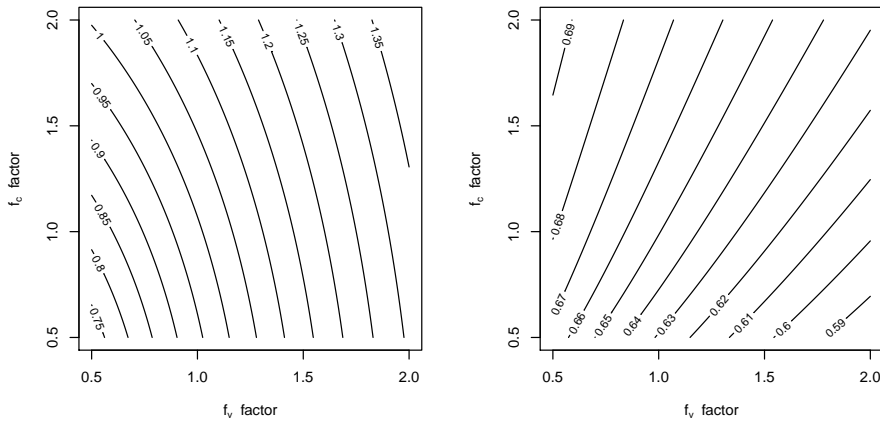
Fig. 3 Evolution of initial population

Fig. 4 Impact of fertility on growth (left) and urbanization (right), example

3.3 Linkage II: location-specific migration probabilities

Only a single migration is possible for each person under Assumption 2 above.
The two Leslie matrices in (4) can also be linked together such that the maxi-
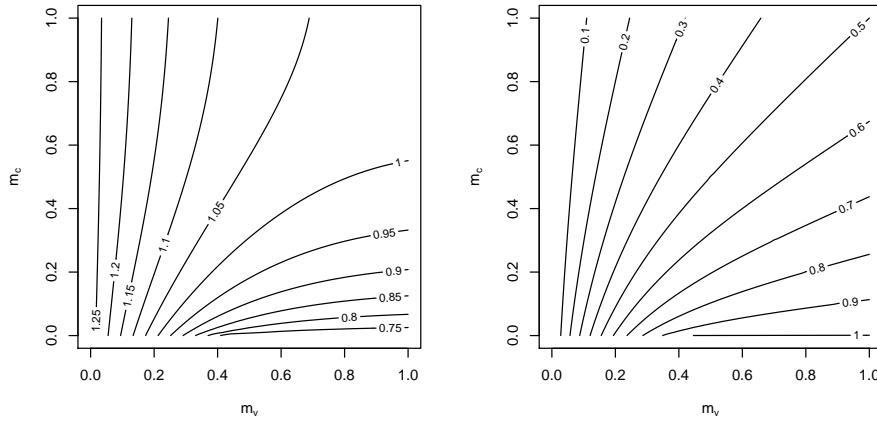mum number of two migrations in a model with three age groups (nine migra-

Fig. 5 Impact of migration on growth (left) and urbanization (right)

tions in a model with ten age groups, see Section 4 below) become possible. One way to achieve this is by replacing Assumption 2 with

– **Assumption 2':** A migrant adopts the migration pattern of her current location:

$$m_{ci}^* = m_{ci}, \quad m_{vi}^* = m_{vi}, \quad i = 1, 2, 3. \tag{10}$$

This amounts to making migration probabilities location-specific in the sense that the status of a potential migrant is irrelevant.

The long-run impact of migration probabilities on growth and urbanization can be studied via the matrix M_I in Table 3, whose structure will now be modified by letting long-run growth and long-run urbanization be a function of $m_v \equiv m_{vi}$ and $m_v \equiv m_{vi}$. In contrast to the procedure when investigating the impact of fertility, we substitute a new set of migration probabilities, m_c and m_v, rather than multiplying by a factor, because a probability is bounded by one. Plots of resulting contour lines are displayed in Figure 5. In particular, a higher city-to-village migration can offset a higher village-to-city migration in terms of equal long-run growth, where the ratio depends on the location of (m_v, m_c). For example, high village-to-city migration will lead to below-replacement overall fertility if city-to-village migration is low (the lower right corner of the left-hand plot in Figure 5). At the same time, urbanization will approach a high level (the lower right corner of the right-hand plot).

4 Analyzing the populations of Germany and Turkey

The goal of this section is to analyze the populations of Germany and Turkey on the basis of the model defined in Section 2. Data concerning age structure, fertility, survival, and migration are presented in the Appendix. The first step will be to give an account of the assumptions made to obtain the projection matrix M_I. This will enable us to compare the actual populations of Germany

and Turkey with their respective stable counterparts, and to discuss the impact of fertility and migration levels on long-run growth and urbanization, similar to the procedure undertaken in Section 3.

4.1 Definition of M_I

The subsequent analysis is based on ten five-year age groups, covering ages 0 to 50. All data used in the definition of M_I are reported in the Appendix. On this basis, we obtain the entries of the projection matrix M_I (see Table 1) as follows:

- Natives' fertility rates f_{ci}, f_{vi}, survival probabilities p_{ci}, p_{vi}, and migration probabilities are taken from Table 7 (Germany) and Table 8 (Turkey).
- Migrants' probabilities to migrate again are obtained as indicated in Assumption 2' in Section 3, that is: migration probabilities m_{ci}^* and m_{vi}^* are location-specific.
- Migrants' survival probabilities p_{ci}^* and p_{vi}^* are obtained as arithmetic means of city native and village native survival probabilities in their respective age class; this is Assumption 3 in Section 3.
- Migrants' fertility rates f_{ci}^* and f_{vi}^* are obtained as arithmetic means of city native and village native fertility rates in their respective age class; this is Assumption 4 in Section 3.

4.2 Actual and stable populations

Given the initial population N_0 and the projection matrix M_I, the model permits a comparison of actual and stable populations in terms of numerical characteristics (Table 4) and in terms of histograms of age distributions in Figures 6 and 7.

	Germany		Turkey	
	actual	stable	actual	stable
urbanization	85.3%	87.5%	65.0%	58.7%
growth, city, annual	-1.0806%	-1.2748%	0.894%	0.302%
growth, village, annual	-1.8154%	-1.2748%	1.239%	0.302%
total growth, annual	-1.1874%	-1.2748%	1.015%	0.302%
median age, city	29.53	28.85	21.81	24.42
median age, village	29.40	28.64	19.75	22.66

Table 4 Characteristics of (female) populations, age 0–50: actual and stable

City and village populations at time t can be obtained by adding native and migrant parts of the vector N_t; urbanization is the share of total population belonging to the city part. Growth measures referring to the actual population result from relating the initial population N_0 to $N_1 = M_I \cdot N_0$; for the stable

population, the maximum eigenvalue of M_I equals the growth factor, and stable growth must be equal for all population parts. The median age is found using linear interpolation of cumulative age group frequencies. Median age here refers to the female population aged 0–50.

In the case of Germany, actual village population shrinks much faster than it would in the case of stability. Median age is very similar in city and village; it is lower in for the stable population. This is confirmed by the age distributions shown in Figure 6: the actual population of Germany is older than its stable counterpart. Frequencies of the stable population in the histograms are essentially increasing, which is a consequence of the population being shrinking, but there is an exception: the slight trough in village population, age group 25–30, is due to migration.
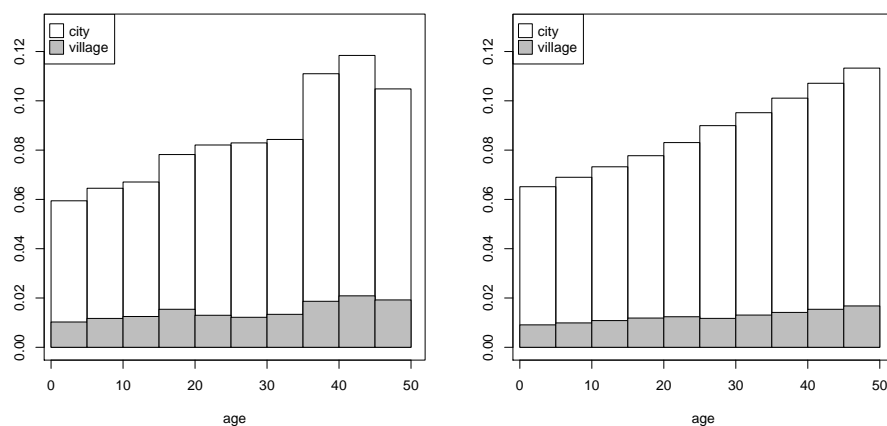

Fig. 6 Actual (left) and stable (right) female population of Germany

The actual Turkish growth figures in Table 4 reflect the young age structure of the population when compared to its stable counterpart, which is also obvious from Figure 7. The relation between city and village with respect to urbanization and median age is the opposite of that in Germany, actual median age being lower than stable median age. The hump (age groups 25–35) in the otherwise monotonically decreasing histogram frequencies in the case of the city age distribution is again due to migration.

4.3 City and village populations in isolation

Partitioning the projection matrix M_I according to (1) reveals the characteristics of city and village population if there were no migration. Projection
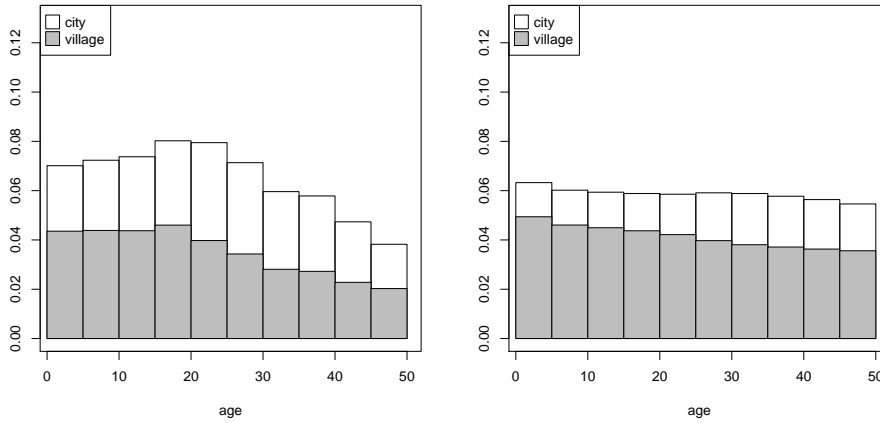
Fig. 7 Actual (left) and stable (right) female population of Turkey

	Germany	Turkey
maximum eigenvalue, $M_{I,\text{city}}$	0.9375	0.9954
maximum eigenvalue, $M_{I,\text{village}}$	0.9412	1.0385
growth, city in isolation, annual	-1.2829%	-0.0922%
growth, village in isolation, annual	-1.2057%	0.7587%
median age, city in isolation	28.85	24.90
median age, village in isolation	28.61	22.30

Table 5 Populations characteristics, age 0–50, without migration

matrices for the populations evolving in isolation are obtained as

$$
\begin{aligned}
M_{I,\text{city}} \quad &= M_I{}^{[\text{city native}]}_{[\text{city native}]} \quad + M_I{}^{[\text{city native}]}_{[\text{village migrant}]}, \\
M_{I,\text{village}} &= M_I{}^{[\text{village native}]}_{[\text{village native}]} + M_I{}^{[\text{village native}]}_{[\text{city migrant}]}.
\end{aligned}
\tag{11}
$$

(Computationally removing migration in this way creates a situation which is similar to the starting point in Section 3.) The projection matrices $M_{I,\text{city}}$ and $M_{I,\text{village}}$ can be analyzed like a classical Leslie matrix. A comparison of the models characteristics in isolation, as given in Table 5, with the characteristics of the full model (Table 4) will then reveal the balancing effect of migration. For example, long-run growth in Turkish cities arises only through migration, while internal fertility is below replacement level. The difference in city and village fertility is much smaller in the case of Germany. The diminished mitigating effect of migration in Germany is also reflected in the smaller difference between city and village median age.

A further interpretation of growth rates of city and village in isolation is that any migration pattern, with which $M_{I,\text{city}}$ and $M_{I,\text{village}}$ are linked together (see Section 3), will lead to an overall stable growth between the values of city and village growth.

4.4 Impact of fertility levels on growth and urbanization

As in Section 3.2, stable growth and urbanization can be plotted as functions of factors with which city fertility $f_c \equiv (f_{c,1}, \ldots, f_{c,10})$ and village fertility $f_v \equiv (f_{v,1}, \ldots, f_{v,10})$ are multiplied. The result is displayed in Figures 8 and 9 where these factors, designated as f_c factor and f_v factor respectively, range from 0.5 to 2.0, corresponding to lowering fertility levels down to 50% and increasing them up to 200% of actually observed levels. We proceed with migrants' fertility according to Assumption 4 above. "Growth" in Figures 8 and 9 refers to five-year intervals.
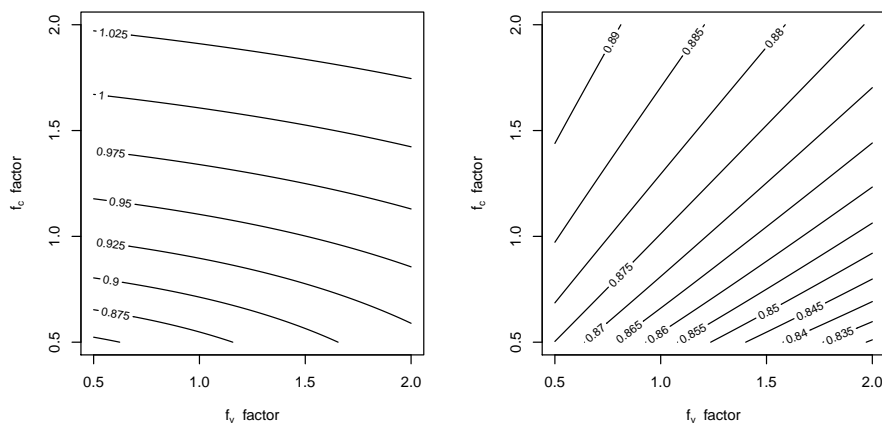
Fig. 8 Impact of fertility levels on growth (left) and urbanization (right), Germany
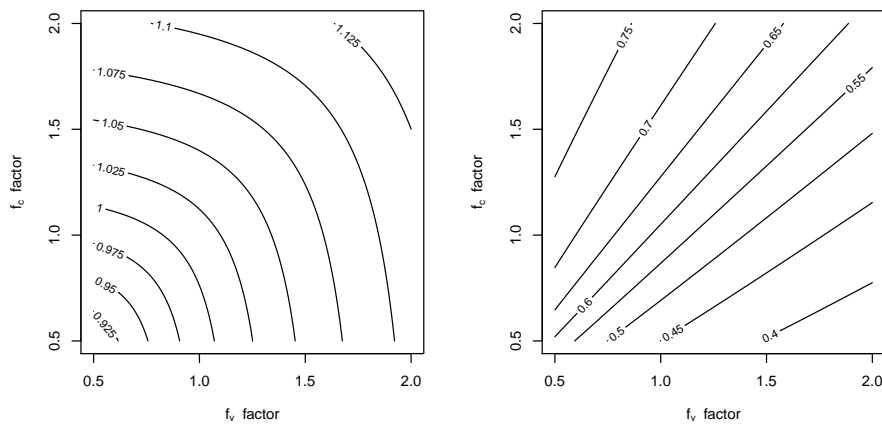
Fig. 9 Impact of fertility levels on growth (left) and urbanization (right), Turkey

There is a substitution effect between city and village fertility, which is more or less constant across the factor range considered for the population in Germany, but not Turkey, where the importance of village fertility increases in a non-linear fashion as city fertility falls below village fertility. This substitution has a big impact on urbanization, with urbanization contour lines standing almost perpendicular to growth contour lines. The shape of urbanization contour lines is very similar for Germany and Turkey.

Fertility levels in Germany are well below replacement level. For example, with village fertility unchanged (f_v factor $= 1$), it can be shown that city fertility would have to be raised by about 61% in order to reach replacement level.

4.5 Impact of migration on growth and urbanization

The impact of migration probabilities m_c and m_v on growth and fertility is shown in Figures 10 and 11. The procedure used to modify the projection matrix M_I is as explained in Section 3.3. As stated earlier, five-year growth will be between that indicated by the maximum eigenvalues of city and village population in isolation (given in Table 5). The impact of the balance between m_v and m_c on urbanization can be grave when both probabilities are small, according to the right-hand plots in Figures 10 and 11. Contour line shapes are similar for the populations of Germany and Turkey, but with growth again at different levels.
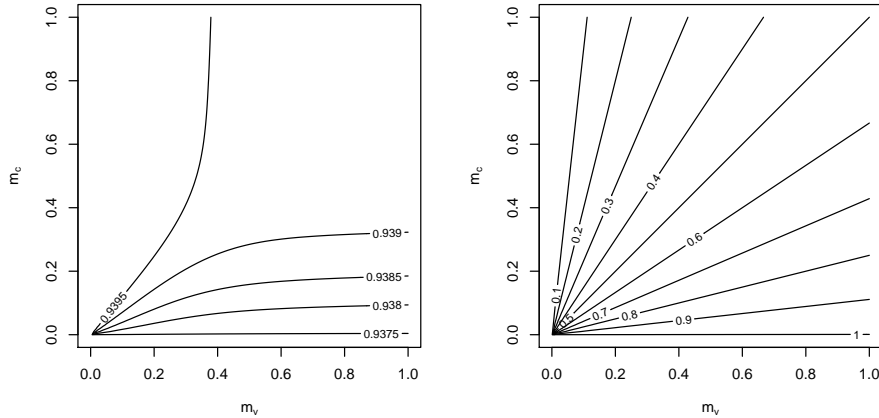

Fig. 10 Impact of migration on growth (left) and urbanization (right), Germany
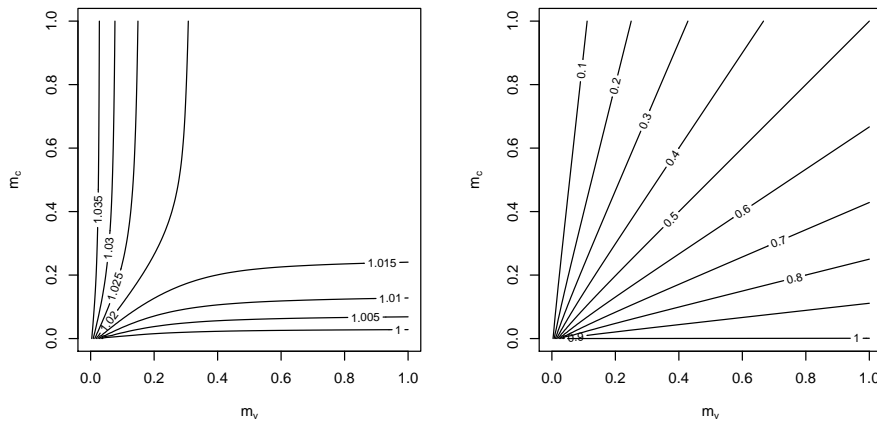
Fig. 11 Impact of migration on growth (left) and urbanization (right), Turkey

5 Summary and conclusions

We use a projection-matrix based population model which takes rural-urban and urban-rural migration explicitly into consideration. Perron-Frobenius theory provides insight into the stable behavior of this model. This approach constitutes a platform for the analysis of the impact of fertility rates, survival probabilities, as well as migration patterns on long-run characteristics of a population. The interplay between fertility, mortality and migration, in particular: how village and city populations are connected together via migration, is illustrated using hypothetical examples.

Applying this model to the female populations of Germany and Turkey, as represented by their respective age structures, fertility rates, and survival and migration probabilities in 5-year age intervals up to age 50, insight could be obtained in two ways: (i) comparing actual population characteristics with their stable counterparts; (ii) analyzing the stable behavior of city and village populations in isolation, when migration is computationally removed from the populations.

In Germany, both actual and stable, city as well as village actual populations are shrinking. The strongest actual negative growth (-1.8% annually) is observed for the village population, which is substantially greater than in the case of the stable growth (-1.3%); the rapid depopulation of rural areas (relative to urban areas) in Germany thus appears as a transient phenomenon. Median age of actual populations is slightly higher than in stable populations: the actual population of Germany is actually older than is would be in the long run, if current demographic parameters persist. Long-run urbanization levels were found to be very similar to its current level, if slightly higher.

Turkish population characteristics differ from those of the German population in almost every respect. Current population growth is positive and higher than in stability ($+0.3\%$ annually), and highest ($+1.2\%$ annually) in the case

of the actual village population. Median age of the city population is markedly higher than of the village population, and stable median age is higher than the actual median age (which explains the currently higher growth rate): the current young population structure of Turkey is, in this sense, a transient phenomenon, and so is the high level of urbanization, which will recede in in the long run if current demographic parameters persist.

When considered in isolation, we find little difference between stable characteristics of city and village populations of Germany, while the stable city population of Turkey was found to shrink, to the effect that the slight overall population increase of Turkey is due to migration. We further find that a slight change in migration patterns can have a profound impact on long-run urbanization in Germany as well as in Turkey.

The framework on which this study is based provides ample opportunity for further research. Two directions are: (i) to elaborate further characteristics of actual populations, and assess the impact of public policies on the future population structure and its economic consequences, such as the age dependency ratio; (ii) to gain further theoretical insight into the model, for example, the transformation of the model into a Markov chain (this was found useful for other Leslie-type models; see, for example, Schmidbauer and Rösch [20]), which would create a platform for the introduction of reproductive values in order to study migration and fertility from a novel perspective.

Appendix: Data for Germany and Turkey

Germany

All data used in our study of Germany are retrieved from the "Regionaldaten-bank Deutschland"[3] and refer to years 2008 and 2009.

Definition of city/village population. The population living in urban areas (the "city population") is the population living outside the rural areas. Our definition of rural areas adopted the OECD urban-rural typology (see [16]). In a first step, local administrative units ("Gemeinden, Samt-/Verbandsgemeinden", LAU2) with a population density below 150 inhabitants per square kilometer were classified as rural. On the next higher territorial level ("Kreise, kreisfreie Städte", NUTS 3 regions), as it provided the data input to our model in the first place, we defined rural areas according to the share of regional population living in rural LAU2 (more than 50%) and the absence of urban centers with more than 200 000 inhabitants accounting for at least 25% of the population. The basic year of this definition was 2008. The population living in such defined rural (respectively predominantly rural)

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[3] http://www.regionalstatistik.de

areas constituted our "village population" in Germany. All further data were aggregated on the basis of this territorial typology.

Table 6 shows the initial (by end of year 2009) female city and village populations of Germany and Turkey (for which an explanation will be given below) used in this study. (The two groups covering ages below 10 were defined by splitting and combining the provided age groups proportionally to lengths.) The share of female village population amounted to 14.5% at that time, with respect to the total female population (i.e. covering all ages).

	Germany		Turkey	
age group	city	village	city	village
0–4	1 434 569	233 617	2 040 448	1 267 489
5–9	1 505 624	264 152	2 105 014	1 276 474
10–14	1 632 282	297 973	2 146 147	1 272 645
15–19	1 779 434	323 606	2 334 348	1 339 118
20–24	2 101 192	316 612	2 312 556	1 156 954
25–29	2 154 056	301 496	2 076 129	998 454
30–34	2 035 725	300 608	1 734 423	817 621
35–39	2 219 078	364 288	1 683 480	793 328
40–44	2 869 289	484 606	1 377 598	663 653
45–49	2 924 688	522 709	1 112 760	590 090

Table 6 The initial (female) age-structured populations of Germany and Turkey

Fertility. Female births during 2009 by age groups of the mother were related to the midyear (average) female population to estimate 1-year fertility rates, and 5-year rates assuming linearity. (Age groups "below 20 years" and "40 years and older" were treated as 5-year intervals.)

Mortality. Probabilities of surviving 5 years from the beginning of each age group are estimated from the age-specific cases of death during 2009 together with a linear approximation of the population at risk at the beginning of each age group (one fifth of the corresponding midyear female population plus half of deaths).

Migration. Migration probabilities from city to village and vice versa, covering 5 years from the beginning of each age group, were obtained in several steps. In the first step, age-specific probabilities of departure from urban (rural) areas are estimated from the cases of departure during 2009. Computations are analogous to the estimation of probabilities of dying within 5 years from the beginning of each age group. As departing from an urban area is not synonymous with arriving to a rural, the share of arrivals to a rural area among all arrivals during 2009 is used to obtain directional migration probabilities. These were further adjusted for internal migration by a factor 0.8321 of non-foreign departures.

The resulting rates and probabilities are given in Table 7.

	fertility rate		survival probability		migration probability	
age group	city	village	city	village	to city	to village
0–4	0.0000	0.0000	0.9964	0.9966	0.1514	0.0247
5–9	0.0000	0.0000	0.9996	0.9996	0.1496	0.0263
10–14	0.0000	0.0000	0.9995	0.9994	0.1488	0.0271
15–19	0.0223	0.0231	0.9991	0.9986	0.2983	0.0443
20–24	0.0951	0.1086	0.9989	0.9984	0.4632	0.0531
25–29	0.1897	0.2299	0.9988	0.9983	0.4154	0.0562
30–34	0.2184	0.2200	0.9982	0.9981	0.1543	0.0235
35–39	0.1084	0.0870	0.9973	0.9971	0.1522	0.0257
40–44	0.0211	0.0154	0.9953	0.9956	0.1517	0.0263
45–49	0.0000	0.0000	0.9917	0.9915	0.1504	0.0277

Table 7 German (female) population data used for the projection matrix M_I

Turkey

Age-specific fertility rates of the urban and rural population in Turkey for the year 2003 are published by the Hacettepe University Institute of Population Studies [7]. — Survival probabilities (year 2000) are taken from WHO [24], abridging the first two intervals. — Substitutes for migration probabilities were obtained as the number of female migrants (village to city or city to village), divided by the total female population in that age group (again village or city, respectively). Both series are published by the Turkish Statistical Institute [22].

Since city- and village-specific survival probabilities were not available, we assume that urban and rural populations have the same survival probabilities. Furthermore, we assume that migrants immediately adopt the destination's (and the natives') migration probabilities, that is, $m_{ci}^* = m_{ci}$, $m_{vi}^* = m_{vi}$. As to fertility rates, we repeat the assumption made in Section **??**: Migrants' age-specific fertility rates are obtained as the arithmetic means of natives of both locations.

Finally, the initial (female) population N_1 of Turkey (for the year 2000; see [22]) is given as shown in Table 6.

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age group	fertility rate		survival probability		migration probability	
	city	village	city	village	to city	to village
0–4	0.0000	0.0000	0.9574	0.9574	0.0500	0.0300
5–9	0.0000	0.0000	0.9969	0.9969	0.0430	0.0284
10–14	0.0000	0.0000	0.9979	0.9979	0.0455	0.0265
15–19	0.1100	0.1180	0.9971	0.9971	0.0631	0.0335
20–24	0.3150	0.4030	0.9962	0.9962	0.1020	0.0445
25–29	0.3150	0.3950	0.9953	0.9953	0.0821	0.0402
30–34	0.1780	0.2350	0.9940	0.9940	0.0531	0.0320
35–39	0.0830	0.1200	0.9918	0.9918	0.0415	0.0272
40–44	0.0280	0.0400	0.9883	0.9883	0.0328	0.0261
45–49	0.0000	0.0150	0.9824	0.9824	0.0264	0.0274

Table 8 Turkish (female) population data used for the projection matrix M_I

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