Analytical Solutions and Approximations to Gompertz-Makeham Life Expectancies

Abstract

We study the Gompertz and Gompertz-Makeham mortality models. We prove that the resulting life expectancy can be expressed in terms of a hypergeometric function if the population is heterogeneous with gamma-distributed individual frailty, or an incomplete gamma function if the study population is homogeneous. We use the properties of hypergeometric and incomplete gamma functions to construct approximations that allow calculating the respective life expectancy with high accuracy and interpreting the impact of model parameters on life expectancy.

Introduction

Parametric models of human mortality date back to Gompertz (1825) and his perception of mortality rates that grow exponentially with age. Makeham's contribution (Makeham 1860) consists in the addition of an age-independant constant that, on the one hand, accounts for mortality that is not related to aging and, on the other hand, introduces an additional third parameter that improves the model fit.

In human populations, the observed overestimation of the Gompertz-Makeham model at ages above 80 demanded the study of models that account for unobserved heterogeneity (Beard 1959). Vaupel et al. (1979) introduced a random variable, called frailty, that modulates individual lifetimes. The resulting mixture of two distributions, one for the general mortality schedule and one for frailty, describes the process at population level. The simplest model (Vaupel et al. 1979) that accurately captures the observed mortality dynamics (Missov and Vaupel 2011) is the Gamma-Gompertz (or Gamma-Gompertz-Makeham) model. Within its framework individual frailty Z is described by a p.d.f.

$$\pi(z) = \frac{\lambda^k}{\Gamma(k)} \, z^{k-1} \, e^{-\lambda z}, \qquad k, \lambda > 0$$

In the simplest model settings frailty is considered to be time-constant, i.e. individuals are born with a certain frailty that remains the same throughout their life. The force of mortality

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and the survival function of an individual with frailty $Z = z$ at age x in year y is given, respectively, by

$$\mu(x, y \,|\, z) = z\, a(y)e^{bx} + c(y) \cdot \mathbb{1}_{c(y)\neq 0} \tag{1}$$

and

$$s(x, y \,|\, z) = \exp\left\{-z\, \frac{a(y)}{b}\left(e^{bx} - 1\right) - c(y)x \cdot \mathbb{1}_{c(y)\neq 0}\right\}, \tag{2}$$

where $a(y) > 0$ is the initial level of mortality, $b > 0$ is the rate of aging, and $c(y) \geq 0$ stands for the level of age-independent extrinsic mortality (Kirkwood 1985). When $c(y) = 0$, $\mu(x, y \,|\, z)$ follows a Gompertz curve. Otherwise $\mu(x, y \,|\, z)$ has a Gompertz-Makeham shape. In our notation we will not further specify the indicator function $\mathbb{1}_{c(y)\neq 0}$, assuming $c(y)$, if present in the formulae, non-zero.

The distribution of lifetimes in a Gamma-Gompertz mixture model is, thus, described by a survival function

$$s(x, y) = \int_0^{\infty} s(x, y \,|\, z)\, \pi(z)dz = e^{-c(y)x}\left(1 + \frac{a}{b\lambda}(e^{bx} - 1)\right)^{-k} \tag{3}$$

Consequently (remaining) life expectancy at age x in year y can be calculated as

$$e(x, y) = \int_x^{\infty} e^{-c(y)t}\left(1 + \frac{a}{b\lambda}(e^{bt} - 1)\right)^{-k} dt \tag{4}$$

The Gompertz(-Makeham) force of mortality $\mu(x, y \,|\, 1)$ is widely used to describe the mortality patterns of not only humans, but also many other animal species (Gavrilov and Gavrilova 1991; Promislow 1991; Golubev 2004; Teriokhin et al. 2004; Bronikowski et al. 2011). A modification of the Gompertz(-Makeham) model, the Logistic or Logistic-Makeham (Pletcher 1999)

$$\mu(x) = \frac{ae^{bx}}{1 + \frac{a}{b}\varkappa\left(e^{bx} - 1\right)} + c, \qquad a > 0, \varkappa \geq 0, \tag{5}$$

is often found to provide a better fit by accounting for the deceleration of mortality, captured by parameter $\varkappa$, at the highest ages (Gotthard et al. 2000; Tu et al. 2002; Fox et al. 2003; Rueppell et al. 2007; Terzibasi et al. 2007).

One way to think about (5) is that the population is homogeneous, and all of the individuals experience the same logistic hazard of death. However, incorporating the idea of heterogeneity, we could interpret (5) as it follows: individual hazard rises exponentially with age, but some individuals are more robust than others and their exponential increases start from different levels. The Logistic-Makeham model is by its parametrization the same as the (heterogeneous) Gamma-Gompertz-Makeham model (Vaupel and Yashin 1985), where different individuals have different levels of frailty and the same exposure to external mortality.

In this article we focus on remaining life expectancy $e(x, y)$ at age x in the Gamma-Gompertz-Makeham model settings and address three questions: i) can we represent $e(x, y)$ and, in particular, life expectancy at birth $e(0, y)$ in an analytical form, ii) if yes, what insight does it give about life expectancy dynamics with respect to the model parameters $a(y), b, c(y), k, \lambda$; iii) would it be possible to derive simple approximations that capture this dependence, on the one hand, and do not destroy the accuracy of approximation, on the other?

Gamma-Gompertz(-Makeham) $e(x, y)$ and Its Approximations

Analytical Solution for $e(x, y)$

Suppose the force of mortality and the survival function of a population are given by (1) and (2), respectively. Then remaining life expectancy at age x in year y is given by (4). Following the idea in the proof of Missov (2010), it can be derived (see Appendix **??**) that

$$
\begin{aligned}
e(x, y) = & \frac{\left(\frac{b\lambda}{a(y)} e^{-bx}\right)^k e^{-cx}}{bk + c(y)} \left(1 - \left(1 - \frac{b\lambda}{a(y)}\right) e^{-bx}\right)^{-k} \\
& {}_2F_1\left(k, 1; k+1+\frac{c(y)}{b}; \frac{\left(1 - \frac{b\lambda}{a(y)}\right) e^{-bx}}{\left(1 - \frac{b\lambda}{a(y)}\right) e^{-bx} - 1}\right),
\end{aligned}
\tag{6}
$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gaussian hypergeometric function, i.e.

$$
{}_2F_1(\alpha, \beta; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j \, (\beta)_j}{(\gamma_j)} \frac{z^j}{j!},
\tag{7}
$$

which is defined for $\gamma > \beta > 0$ (see, for example, Bailey 1935); $(m)_n = m(m+1) \ldots (m+n-1)$ denotes the Pochhammer symbol.

Eq. (6) reduces for $x = 0$ to the Gamma-Gompertz-Makeham life expectancy at birth

$$
e(0, y) = \frac{1}{bk + c(y)} \, {}_2F_1\left(k; 1; k+1; 1 - \frac{a(y)}{b\lambda}\right),
\tag{8}
$$

which is equal to the expression for Gamma-Gompertz life expectancy, derived by Missov (2010), for $c(y) = 0$.

Approximations for $e(x, y)$

First, we will use a linear transformation of the hypergeometric function and some of its properties (Abramowitz and Stegun 1965:15.3.4,15.3.8,15.1.8), i.e. for $z = e^{-bx}\left(1 - \frac{b\lambda}{a(y)}\right)$ we have

$$
{}_2F_1\left(k; k + \frac{c(y)}{b}; 1 + k + \frac{c(y)}{b}; z\right) = (1 - z)^{-k}\left(\frac{bk}{c(y)} + 1\right){}_2F_1\left(k, 1, 1 - \frac{c(y)}{b}, \frac{1}{1 - z}\right) +
$$
$$
+ (-z)^{-k - \frac{c(y)}{b}}\frac{\Gamma\left(1 + k + \frac{c(y)}{b}\right)\Gamma\left(-\frac{c(y)}{b}\right)}{\Gamma(k)} .
$$
(9)

The z-argument of the hypergeometric function in the first additive term on the right-hand side of (9) approaches 0 (for the given human-mortality parameter values above). This implies that the general term of the series (7) will tend to 0 and we can use the first two terms of the hypergeometric series on the right-hand side of (9) as its approximation:

$$
{}_2F_1\left(k; k + \frac{c(y)}{b}; k + 1 + \frac{c(y)}{b}; z\right) \approx (1 - z)^{-k}\left(\frac{bk}{c(y)} + 1\right)\left(1 + \frac{k}{1 - \frac{c(y)}{b}}\frac{1}{1 - z}\right) +
$$
$$
+ (-z)^{-k - \frac{c(y)}{b}}\frac{\Gamma\left(k + 1 + \frac{c(y)}{b}\right)\Gamma\left(-\frac{c(y)}{b}\right)}{\Gamma(k)} .
$$

As a result, the corresponding approximation for $e(x, y)$, which contains also the multiplicative terms in the right-hand side of (8), for $z = e^{-bx}\left(1 - \frac{b\lambda}{a(y)}\right)$ will be given by:

$$
e(x, y) \approx \frac{\left(\frac{b\lambda}{a(y)}e^{-bx}\right)^k e^{-cx}}{bk + c(y)}\left[(1 - z)^{-k}\left(\frac{bk}{c(y)} + 1\right)\left(1 + \frac{k}{1 - \frac{c(y)}{b}}\frac{1}{1 - z}\right)\right.
$$
$$
\left. + (-z)^{-k - \frac{c(y)}{b}}\frac{\Gamma\left(k + 1 + \frac{c(y)}{b}\right)\Gamma\left(-\frac{c(y)}{b}\right)}{\Gamma(k)}\right]
$$
(10)

or when $a(y)$ is close to 0 and $(1 - z) \approx (-z)$ and also $1/(1 - z) \approx 0$,

$$e(x,y) \approx \frac{\left[\left(1 - \frac{a}{bk}\right)e^{\frac{c(y)}{k}x}\right]^{-k}}{bk + c(y)} \left[\left(\frac{bk}{c(y)} + 1\right) + (-z)^{-\frac{c(y)}{b}} \frac{\Gamma\left(1 + k + \frac{c(y)}{b}\right)\Gamma\left(-\frac{c(y)}{b}\right)}{\Gamma(k)}\right] \quad (11)$$

Example

In order to illustrate the accuracy of approximation, we fitted a Gamma-Gompertz-Makeham force of mortality to the 2007 United States data, the 2009 Japan data, the 2009 Germany data, and the 2010 Sweden data (ages 30 to 110) with Poisson likelihood estimation. The resulting parameter estimates are presented in the following table:

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
$\hat{a}_{30}$	8.68E-05	1.65E-04	1.45E-04	3.49E-04
$\hat{b}$	0.127	0.117	0.112	0.101
$\hat{\gamma}$	0.100	0.097	0.093	0.110
$\hat{c}$	0.0005	0.0007	0.0006	0.0010

Table 1: Gamma-Gompertz-Makeham parameter estimates

$\gamma = k/\lambda^2$ denotes the variance of the gamma distribution. Using (6) for the exact result, (10) for the approximate value, and (11) for a cruder approximation, we get the following life-expectancy values

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
exact	52.380	50.824	53.825	49.697
approx$_1$	52.379	50.823	53.824	49.691
approx$_2$	52.375	50.812	53.814	49.663

Table 2: Exact and approximate values for remaining Gamma-Gompertz-Makeham life expectancy at age 30

Gompertz(-Makeham) Life Expectancy and Its Approximations

In this section we derive an analytical solution for life expectancy in a homogeneous population, assuming that the individual force of mortality follows the Gompertz(-Makeham) law, i.e. $\mu(x, y \,|\, 1)$ (see eq. 1).

The Gompertz-Makeham Life Expectancy and Its Approximation

Life expectancy at age x in year y can be given explicitly by

$$e_{GM}(x,y) = \frac{1}{b} e^{\frac{a(y)}{b}} \left(\frac{a(y)}{b} \right)^{\frac{c(y)}{b}} \Gamma \left(-\frac{c(y)}{b}, \frac{a(y)}{b} e^{bx} \right), \tag{12}$$

where $\Gamma(s, z) = \int\limits_z^\infty t^{s-1} e^{-t} dt$ denotes the upper incomplete gamma function (see **??** in Appendix).

If a is close to 0, $e_{GM}(0, y)$, can be approximated by

$$e_{GM}(0, y) = \frac{1}{c} - \frac{\left(\frac{a}{b} e^{\gamma - 1} \right)^{\frac{c}{b}}}{c(1 - \frac{c}{b})}, \tag{13}$$

where $\gamma \approx 0.57722$ is the Euler-Mascheroni constant.

For credible extreme values of human mortality, $0 < \frac{a(y)}{b} e^{bx} \le 1$ and $0 < \frac{c(y)}{b} \le 0.1$, $\Gamma \left(-\frac{c(y)}{b}, \frac{a(y)}{b} e^{bx} \right)$ can be approximated by (see **??** in Appendix)

$$\Gamma(s, z) = \frac{1}{s + s^2} \exp \left\{ (1 - \gamma)s + 0.3225 s^2 \right\} - \sum_{k=0}^\infty (-1)^k \frac{z^{s+k}}{k!(s+k)}, \tag{14}$$

where $\zeta(n) = \sum\limits_{k=1}^\infty k^{-n}$ is the Riemann zeta function and $0.3225 \approx \frac{\zeta(2)-1}{2}$. The closer the z-argument of the upper incomplete gamma function to 0 is, i.e. at younger ages, the less terms of $\sum\limits_{k=0}^\infty \frac{(-1)^{k+1} z^{s+k}}{k!(s+k)}$ we need to use. The number of terms m, which have to be taken in the latter series into account to achieve a desired accuracy ε, can be determined by

$$\frac{z^{s+m+1}}{(m+1)!(s+m+1)} \le \varepsilon.$$

Example

Fitting a Gompertz-Makeham model by Poisson maximum likelihood for the 2007 United States data (ages 30 and above), we get the following parameter values: $\hat{a}_{30} = 0.00046$, $\hat{b} = 0.094$ and $\hat{c} = 0.0007$. If we want to measure remaining life expectancy at age 30, by calculating $\hat{s} = -0.0074$ and $\hat{z} = 0.0049$ from the fitted parameters and setting the error of approximation to the sum by $\varepsilon = 0.001$,

$$\frac{0.0049^{-0.0074+m+1}}{(m+1)!(-0.0074+m+1)} \leq 0.001$$

solving this inequality for m gives, $m = 0.24$ or rounding up to the first integer as only integers are allowed for k in (14), yields $m = 1$. Similarly, when the Makeham term is close to zero, in the exponential part of (14) the second, $\zeta(n)$ zeta function term can be left out of the approximation. In this case, it would cause an approximation error of 0.0014. Adding the two errors together, by using

$$\Gamma(s, z) = \frac{e^{(1-\gamma)s}}{s+s^2} - \frac{z^s}{s} + \frac{z^{s+1}}{s+1} \tag{15}$$

instead of Eq. 14 approximates $\Gamma(s, z)$ with an error of less than 0.0024.

Finally, calculating the remaining life expectancy at age 30 for the United States in 2007, Japan and Germany and 2009, as well as Sweden 2010 by Eq. 12, and by substituting Eq. 15 in Eq. 12 gives

	Sweden 2010	Germany 2009	Japan 2009	USA 2007
exact	51.69	49.97	52.91	48.42
approx	51.70	49.99	52.92	48.44

Table 3: Exact and approximate values for remaining Gompertz-Makeham life expectancy at age 30

respectively. Please note that the approximation error of (15) is inflated by the multiplicative terms preceding the upper incomplete gamma function in (12) leading to the error of not more than 0.02.

Gompertz Life Expectancy and Its Approximation

Without the Makeham term, for the proportionally changing Gompertz force of mortality used by Vaupel (1986)

$$\mu_G(x, y) = a(y)e^{bx},$$

Missov and Lenart (2011) showed that the corresponding life expectancy of age x in year y can be solved as

$$e_G(x, y) = \frac{1}{b} e^{\frac{a(y)}{b}} E_1\left(\frac{a(y)}{b} e^{bx}\right),$$

where $E_1(z) = \int\limits_z^\infty \frac{e^{-t}}{t} dt$ denotes the exponential integral. Note that

$$E_1(z) = \lim_{s \to 0} \int\limits_z^\infty t^{s-1} e^{-t} dt = \lim_{s \to 0} \Gamma(s, z),$$

i.e. $e_G(x, y)$ is a degenerate form of $e_{GM}(x, y)$ when the Makeham term equals 0.

As (Abramowitz and Stegun 1965:5.1.11)

$$E_1(t) = -\gamma - \ln t - \sum_{n=1}^\infty \frac{(-1)^n \, t^n}{n \cdot n!},$$

if $t = \frac{a(y)}{b} e^{bx}$ is close to 0, then $e_G(x, y)$ can be approximated by

$$e_G(x) \approx \frac{1}{b} e^{\frac{a(y)}{b}} \left(-\gamma - \ln \frac{a(y)}{b}\right). \tag{16}$$

Connecting Eq. 15 to Eq. 16, i.e. when $c(y) = 0$, we get that

$$\Gamma\left(0, \frac{a(y)}{b} e^{bx}\right) = \lim_{s \to 0} \left\{\frac{e^{(1-\gamma)s}}{s + s^2} - \frac{z^s}{s} + \dots\right\} = \lim_{s \to 0} \left\{\frac{e^{(1-\gamma)s} - z^s(1 + s)}{s + s^2} + \dots\right\}$$

Applying l'Hôpital's rule, gives

$$\Gamma\left(0, \frac{a(y)}{b} e^{bx}\right) = \lim_{s \to 0} \left\{\frac{(1 - \gamma)e^{(1-\gamma)s} - (1 + s)z^s \ln z - z^s}{1 + 2s} + \dots\right\} = -\gamma - \ln z + \dots. \tag{17}$$

Substituting back Eq. 17 to Eq. 12 shows that Eq. 15 leads to the same result as the Gompertz approximation when $c(y) = 0$.

Elasticities

In this section we calculate the elasticities of life expectancy with respect to the Gompertz-Makeham parameters, i.e. the ratio of the relative change in life expectancy to the relative

change in a, b, and c. We use the definition of elasticity in economics, i.e. the elasticity $E_{y,x}$ of y with respect to x is given by

$$E_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y}$$

Calculating the elasticities of Eq. 13 gives

$$E_{e_{GM},a} = \frac{\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}}{1 + \frac{b}{c}\left[\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}} - 1\right]}$$

with respect to a.

$$E_{e_{GM},b} = \frac{\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}\left[c - 2b + (c-b)\ln\left(\frac{a}{b}e^{\gamma-1}\right)\right]}{\left(\frac{b}{c} - 1\right)\left[c + b\left(\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}} - 1\right)\right]}$$

with respect to b.

$$E_{e_{GM},c} = \frac{c^2 + b^2 - 2bc + \left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}\left[-b^2 + 2bc + c(b-c)\ln\left(\frac{a}{b}e^{\gamma-1}\right)\right]}{(b-c)\left[c + b\left(\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}} - 1\right)\right]}$$

with respect to c.

Substituting $a = 0.00002$, $b = 0.1$ and $c = 0.0007$ in the equations above for modern societies and $a = 0.000004$, $b = 0.125$, $c = 0.011$ for hunter-gatherers (Gurven and Kaplan 2007) show that life expectancy is highly inelastic to small changes in a or c and more sensitive to changes in b.

Elasticity of	Modern society	Hunter-gatherer
a	-0.12	-0.07
b	-0.85	-0.575
c	-0.028	-0.355
k	-0.13	
λ	0	
k*	-0.01	

Table 4: Elasticities of Gompertz-Makeham parameters

That is, in case of modern societies, a 10% increase in a would decrease the life expectancy by 1.2%, a 10% increase in b would decrease the life expectancy by 8.5% and a 10% increase in c would decrease life expectancy by 0.03%. Increasing shape parameter of the distribution of heterogeneity decreases the proportion of lower frailty individuals and in accordance with it, the elasticities show that if k increases by 10%, life expectancy would decrease by 1.3%. Note that in this case, the average frailty of the population would increase. An increase in the λ, the scale parameter of the frailty distribution would have 0 or very mild positive effect on life expectancy as it decreases the mean frailty. We denoted by k^* the situation when $k = \lambda$ and the mean frailty therefore equals to 1. In this case, an increase in k^* would only mean that the heterogeneity of the population decreases and the mean of it stays the same. As the population is getting more homogeneous (with the same mean frailty value), the life expectancy of the population mildly decreases, a 10% increase in k^* would decrease life expectancy by 0.1%. However, please note that the elasticities are not constant; for a higher value of a or c, life expectancy can be more elastic as the case of the hunter-gatherers show.

A simpler representation of the analytic forms of the elasticities is possible by approximating them by Chebyshev polynomials of the first kind. A second degree polynomial approximation shows the sensitivity of the elasticities to changes in c (Table 5). Substituting a value for c or a in these formulas gives the elasticity of life expectancy with respect to any of its parameters. For example, looking at the row of b, we can see that if c increases, a change in b loses its importance in determining the life expectancy. Nevertheless, for $c \approx 0$, a change in the rate of aging would have the largest impact on life expectancy.

Elasticity	in terms of c	
w.r.t	homogeneous	heterogeneous
a	$-0.12 + 5c - 5.62c^2$	$-0.13 + 5c - 5.6c^2$
b	$-0.88 + 45.65c - 48.07c^2$	$-0.87 + 58.22c - 66.07c^2$
c	$-47.65c + 53.69c^2$	$-63.22c + 71.67c^2$
k		$-0.14 + 5.22c - 5.84c^2$
λ		$0.87c - c^2$
k^*		$-0.01 + 0.22c - 0.24c^2$

Table 5: Elasticities of Gompertz-Makeham parameters as Chebyshev polynomials

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