Childbearing and the Labor Market: Time and Space Dynamics

Elena Kotyrlo
PhD-student, Umeå School of Business and Economics, Umeå University[1]
SE-901 87, Umeå, Sweden

Fertility is an important determinant of long-run population growth and labor market conditions. The present study focuses on the effects of time and space dynamics on the description of fertility in Sweden. These effects were expected to be generated by labor mobility across municipalities. The influence of time dynamics in postponing or accelerating childbearing was assessed by considering two different effects of earnings. Firstly, the effect within one generation was considered by comparing a family's current earnings with their earnings in the recent past and expected earnings in the future. The second effect, referred to previously as the Easterlin hypothesis, was examined through the generations by comparing a household's earnings for a younger generation with earnings of the parental generation. The hypotheses were tested for the period 1981-2008. The study involved estimating space and time dynamics by using the SAR(2,1) model and the general method of moments for aggregate panel data. By comparing different specifications, positive spatial autocorrelation of fertility was identified. Current earnings appeared to have a negative effect on fertility rates within municipalities, and in the long-run, across them. The inverted Easterlin hypothesis was weakly supported within municipalities. The study makes an important theoretical contribution through the application of a stationarity condition and evaluation of the long-run effect in the direct, indirect and total forms of the SAR (2,1) model with second-order autoregressive and first-order spatial disturbances.

[1] Acknowledgments. I would like to thank my supervisor and co-supervisor, Magnus Wikström and Niklas Hanes, who directed me in this research. My sincere gratitude to Kurt Brännäs, who helped me with application of spatial econometrics. I appreciate Kirk Scott and other senior researchers and participants of the conference for Ph.D. students in economic demography in Mölle, Sweden, 2010, for valuable comments on earlier version of the paper. I also want to thank to participants of the 1st Nordic Winter Conference in Hemavan, Sweden, 2011, 6th International Conference on Population Geographies in Umeå, Sweden, 2011 and to participants of the Applied Demography Conference in San Antonio, TX, 2012, for comments to my presentations of the research. Aknowledged with gratitude for its financial support is the Graduate School in Population Dynamics and Public Policy at Umeå University.

1 Introduction

During the 20th century, a dramatic decline in fertility was observed in developed countries, which was associated with the introduction of contraceptives and increased female labor market participation. Despite the overall downward trend, the variation in total fertility rates (TFR) differs from one country to another. Fertility varies as a consequence of economic events and changes in the cost of raising children. In this paper, fertility time and space dynamics in Sweden are analyzed, particularly in relation to the labor market.

The short-run effects of these factors can be assessed by examining two effects of earnings, in which children are regarded as "normal goods" in the maximization of household utility (Becker and Barro, 1988; Becker, 1960). *The income effect* assumes that a larger income encourages families to have more children. Childbearing can be viewed as "household production" and is considered *a substitute* for female labor market participation. Thus, higher wages raise the opportunity cost of having a child.

Current income and changes in expected earnings contribute to the effect of earnings, implying that a higher income helps a family to deal with the direct costs of childbearing. Thus, in the present study, we anticipated that the postponement or acceleration of childbearing occurs in the short-run in Sweden because the Swedish completed cohort fertility rate is rather constant, but the variation in total fertility rate is about 10% higher than in other Nordic countries during the studied period.

In the long-run, the age structure of the population may affect the total fertility rates and economic explanation of this phenomenon. This principle, known as the Easterlin hypothesis (Easterlin, 1966), relates to a negative response of fertility to increasing labor market tightness caused by increasing numbers of people of working age. According to this hypothesis, decisions about childbearing, which presumably primarily concern the young generation, are made on the basis of the income potentials of the young generation compared to previous generations.

Labor mobility, which has exhibited a positive trend over the last few decades in Sweden (Eliasson et al., 2007), contributes to the space dynamics of fertility. Growing flows of in- and out-migrated people increase the probability of finding a partner, which may lead to marriage or cohabitation, and having a child. In addition, these labor flows generate earning flows across space and affect families' income potentials for childbearing.

New econometric instruments, such as spatial econometrics for panel data, allow a more in depth study of these problems than previously possible, by taking into account the diffusion of fertility norms and the influence of economic factors across space. Municipalities can be considered as open demo-economic systems. Interactions between adjacent municipal labor markets, resulting from the flow of in- and out-migrated people, affect the labor market equilibrium as well as the spatial income distribution. In particular, a gain in local labor market tightness, caused by the entry of a large young generation, can be smoothed by increasing labor mobility, but it would worsen the situation in surrounding labor markets. Thus, the inverted Easterlin hypothesis is expected to be supported in the spatial dimension when considering income potential coupled with tightness of surrounding labor markets; increasing tightness reduces peoples' opportunities to find a good job or compete for higher earnings.

The spatial effect of earnings can be explained in terms of the influence of the average annual income per capita in surrounding municipalities on the total fertility rate in a certain municipality. This includes the effect of migrants, whose earnings are considered statistically to be in the place (municipality) of work, despite the fact that they may have their families in other municipalities.

One important study which measured fertility in the spatial dimension is that of Waldorf and Franklin (2002), who tested the Easterlin hypothesis by assuming the fertility among 18 Italian regions was governed by spatial diffusion. Two types of spatial diffusion between units are considered, namely space interaction of fertility norms and labor mobility, which both influence fertility.

The purpose of the present work is to study how the labor market situation, as measured by households' earnings, influences fertility, taking into account dynamics in time and space. Space diffusion is assessed by global and local spillover effects. Transition of fertility norms across municipalities gives a first-order spatial autocorrelation. Influence of the relative cohort sizes in surrounding municipalities on fertility norms in a given one and cross-municipal influence of the space diffusion of income generated by labor mobility are assumed to be a local form of spillover. Time dynamics are examined by considering a second-order serial autocorrelation of total fertility rates and testing the direct effect of the Easterlin hypothesis and the effect of current earnings in a given municipality. Measurement of space diffusion is based on two types of weight matrices to avoid multi-correlation among the explanatory variables. The row-standardized contiguity weight matrix is used for lagged total fertility rates, whereas matrices with spherical distances weighted by population size are applied for estimating the influence of average annual income per capita and relative cohort size on fertility rates.

The main contribution of this paper is the analysis of spatial interdependence of fertility in an open municipal demo-economic system using a panel of data. In contrast to previous studies, the present work considers the Easterlin hypothesis in terms of the long-run impact of earnings and, at the same time, assesses the relationship between current earnings and fertility. The use of panel data allows space diffusion of fertility norms to be monitored as a function of time in “three dimensions”. The effects of earnings on fertility are related to factors such as skill characteristics of labor markets and labor flows. In contrast to other papers, municipal level data is analyzed, which gives more detailed results than studies at national or regional levels. The theoretical contribution of the paper is in the application of stationarity conditions in the SAR(2,1) model and the derivation of the long-run effects of the explanatory variables in direct, spatial and total forms.

The paper is structured as follows. Section 2 describes the main econometric methods for dynamic models of space and time; previous studies of fertility in spatial diffusion; studies of the Easterlin hypothesis for Sweden; and results concerning the impact of earnings on fertility in Sweden. Section 3 explains the methods used for estimation and post-estimation interpretation, e.g., to evaluate the short- and long-run effects within and across municipalities as well as total effects. A stationarity condition for the estimated model is also discussed. Section 4 describes the data set. Section 5 details the empirical specification. In section 6, the results are presented and then summarized in the final section.

2 Previous Literature

Spatial econometrics of panel data has gained increasing popularity as a new rapidly growing branch of econometrics, since in many cases, the importance of nearest neighbors on social and economic behavior and activity seems self-evident. The derivation of the estimated model, approach to estimation and post-estimation analysis, as well as interpretation of the model and results, have been detailed in earlier papers by Anselin (2002), Brueckner (2003), Elhorst (2001), LeSage and Pace (2010), and Yua and Jong (2008). Brueckner (2003) classified spatial interactions according to a spillover model, whereby spatial units reciprocally affect each other, or a resource-flow model, in which they share some limited resources. Labor mobility may be considered from both points of view. It exhibits a spillover effect when labor mobility increases matching of couples, and consequently, fertility. It can also be interpreted as a model of common resource sharing, such as a total earning potential of the labor market. It is useful that, analytically, both models give rise to a spatially lagged econometric specification (Anselin, 2002, p.250), since it is not possible to attribute labor mobility to purely one effect.

Elhorst (2001) has provided a thorough analysis of first-order autoregressive panel data models in both space and time, including a taxonomy of the models, approaches for estimation and determining stationarity in the time conditions, and a spatial equilibrium correction model, which provides the static long-run equilibrium relationship between endogenous and explanatory variables.

The paper of Yua and Jong (2008) gives a better understanding of the general approach for employing stationary time conditions in spatial modeling. LeSage and Pace (2010) have provided an interpretation of the direct, the indirect (spatial) and total effects, which has been accepted by other researchers as a standard approach for spatial models. The interpretation can be problematic because the indirect effect is specific for each pair of spatial units and the size of the effects matrix depends on the number of units. Thus, LeSage and Pace (2010) suggested measuring the spatial effect compared to the direct and total effects. The average direct effect is interpreted as a mean of diagonal elements. The indirect effect for unit i of a variable x is defined as the sum of off-diagonal elements j=1,...,N of row i and the average indirect effect is the mean of i=1,...,N indirect effects. The average total effect is calculated as the sum of the average direct and indirect effects.

Several studies have considered spatial diffusion of fertility from the perspective of spreading knowledge about contraception (Bongaarts and Watkins, 1996; Weeks et al., 2004; Woods, 1984). These papers have mainly focused on developing countries or historically remote time periods of developed countries. There are substantially fewer papers reporting investigations of space dynamics of fertility in developed countries (de Castro, 2007, McNicoll, 1980, Waldorf and Franklin, 2002).

The Easterlin hypothesis for cross-sectional and panel data is a popular approach for interpreting the effects of earnings on fertility. Macunovich (1998) reviewed 185 published articles incorporating 76 empirical analyses of the Easterlin hypothesis and concluded that the results were mixed. Such ambiguous results may be due to testing the Easterlin hypothesis in the absence of other controls and the assumption that households' incomes and male earnings are interchangeable. Waldorf and Byun (2005) performed a meta-analysis of 334 empirical papers to test the Easterlin hypothesis, which showed a more robust negative effect, despite positive effects being more frequent. A negative effect or inverted Easterlin hypothesis implies that a more numerous young generation than parental generation is related to a higher total fertility rate, when the young generation is of fertility age.

The importance of the relative cohort size, *R*, as an indicator of relative economic status depends on the age range defining the young generation. Researchers have varied the upper boundary from 29 to 34 years in order to maximize the correlation between fertility and age structure. Waldorf and Byun (2005) concluded that the use of a broad age range for the young cohort increases the likelihood of a negative correlation.

Several cross-sectional studies have tested the Easterlin hypothesis in different countries, such as Artzrouni and Easterlin (1982), Baird (1987), Pampel (1993), Wright (1989) and Sevilla (2007). Most of these papers (except Artzrouni and Easterlin (1982) and Baird (1987)) have reported that the inverted Easterlin hypothesis, i.e., a positive relationship between fertility and the proportion of the young generation, applies for Sweden.

Pampel (1993) concluded that the institutional structure, such as family policy, together with increased female labor force participation, influence the relative economic status of the cohort, which explains the negative effect of the cohort size. Hence, the relative cohort size positively influences fertility only when it is associated with poor opportunities for employment, higher wages, and promotion. Pampel proposed that state policies to keep unemployment at a minimum and guarantee jobs, as well as policies against sex discrimination in the labor market, an effective child-care system and the length of maternity leave helped to explain the insignificance of the cohort size effect on fertility. This conclusion is important for understanding the effect of earnings on fertility in Sweden, where family policy plays a major role in reducing the costs associated with raising children. The substitution effect on childbearing is weakened in Sweden not only because maternity benefits are closely related to woman's pre-birth earnings, encouraging women to be employed before childbearing, but also women are able to return to the labor market despite having small children due to the supportive child-care system. Studies based on Swedish micro-data support the positive effect of earnings on fertility (Andersson, 1999; Hoem, 2000).

The skill level of the labor force also affects the studied relationships. On the one hand, women who achieve higher levels of education generally receive higher salaries, which increase the opportunity costs of childbearing and, consequently, reduce the demand on children. Thus, the proportion of high- and low-skilled women employed in a local economy will affect the fertility rate. On the other hand, Andersson et al. (2003) found that women with university education have a higher probability of having a third birth than women with lower education levels. This may in part be due to the family policy in Sweden, where women with a university education generally have better opportunities than others to combine work and family as a result of higher salaries and more flexible working hours. Education and childbearing have been shown to be highly related from a household productivity or investment point of view (Becker and Barro, 1988). Thus, the status of university cities is usually controlled in regression analyses of fertility (Andersson et al., 2003; Westerberg, 2006).

3 The research strategy

Three different hypotheses are tested in this paper. The first is the Easterlin hypothesis, which assumes that the dynamics of age structure affect the tightness of the labor market and considers the role of income potential of the young generation relative to the older generation in explaining decisions concerning childbearing. The cohort ratio is the main explanatory variable. The second hypothesis concerns the short-run effect of earnings, in which the present income potential of households is compared to their earnings in the recent past and expected earnings in the future. This hypothesis predicts an increase in total fertility rates during macroeconomic boom-periods or better family policy regimes and local labor market conditions. The main explanatory variable considered in the second hypothesis is income. The third hypothesis concerns spatial effects on the relationship between fertility and age structure and between fertility and current earnings. The existence of autocorrelation of fertility rates across municipalities, generated by labor movements and “earnings’ flow”, is presumed. The hypotheses are considered assuming the spatial diffusion of fertility rates. The endogenous variable in the model is the annual total fertility rate within a certain municipality.

In the assumption of employing only conditional spatial terms in the model our research strategy is based on the Arellano-Bond linear dynamic panel-data estimation for LAG type of spatial modeling, instead of the maximum likelihood method. This assumption is reasonable since childbearing is a lagged process in relation to factors affecting decisions concerning fertility. The model is estimated for different sets of municipal variables as discussed further below.

Spatial reaction functions can be incorporated into a lag model as either a global form of spillovers or spatial autocorrelation, or as a local form of spillovers, using spatial lag as an explanatory variable. Global and local form spillovers can be combined in one model, but in such a case, the problem of multicollinearity arises, since exogenous variables are included explicitly in a local form and implicitly in a global form. To avoid this problem, distinct weight matrices can be used for lagged-dependent and explanatory variables.

The model, in which fertility has a spatio-temporal lag structure and the spatial influence of age structure and income are extracted, has a linear form. The weighted lagged total fertility rate (**W·TFR**) reflects the recent space-autoregressive dependence of fertility rates due to the possible balancing of age-gender inequality and interactions between people living in different municipalities who potentially form matches during work and leisure. Even though both the cohort ratio R and income I are implicitly included in the specification via the global form of spillover

W·TFR, their spatial influence may be extracted by incorporating local-form spillovers or spatially weighted variables **V·R** and **V·I**.

Several types of matrices for summarizing the spatial morphology of fertility across municipalities are empirically tested on the best approximation of diffusion of fertility rates, using incomes and cohort ratios as the main explanatory variables. As a result of this test, two approaches suggested by Waldorf and Franklin (2002) and based on a matrix of contiguity **W** and a set of matrices **V**(t) with spherical distances between municipalities weighted by population, are chosen for use in the model. These approaches are convenient for reducing multicollinearity and in the latter case, taking account of the influence of larger (in terms of population size) cities.

The row-standardized matrix W is constructed under the assumption that fertility is a spatial stationary process, where the covariation of fertility rates in two municipalities is purely a function of the distance between them. Here, $w_{ij} = 1/k_i$, if municipalities defined by indices i and j ($i, j = 1,..., N$) share a common border, where k_i is the number of municipalities bordering i and N is the total number of municipalities, otherwise $w_{ij} = 0$.

Row-standardized weight matrices **V**(t) based on spherical distances between municipalities are used to summarize the spatial morphology due to the influence of cohort ratio and incomes on fertility:

$$v_{ij}(t) = \frac{Pop_j(t)/d_{ij}}{\sum_k Pop_k(t)/d_{ik}} \qquad (1)$$

where d_{ij} is the spherical distance between municipalities identified by the indices i and j. A standard approach (Anselin, 2002) was employed to construct the weight matrix for the situation where a variable potentially affects the spatial interaction, i.e., in our case, the population size of the municipalities. In a panel data analysis, larger cities cannot be controlled directly by using a dummy variable or incorporating population size (approximately constant for a given unit) into the model. The set of v_{ij}, j=1,...,N for a given i allows higher weights to be ascribed to larger cities, and thus dynamics associated with the size of the municipality are captured in the weight coefficients.

The corresponding panel data matrices have the form $\mathbf{W}_{NT} = \mathbf{I}_T \otimes \mathbf{W}_N$ [2] and $\mathbf{V}_{NT} = \mathbf{I}_T \otimes \mathbf{V}_N(t)$, t = 1,...,T, where T = 28 for the period 1981-2008, and $\mathbf{I}_T$ is an identity matrix of dimension T.

4 Data

Municipality level data for 276 municipalities over the period 1981-2008 are included in the analysis (Statistics of Sweden)[3]. A description of variables and

---

[2] $\otimes$ is a Kronecker product

[3] The table A3 provides some information about constructing data with respect to restructuring of municipalities in 1992.

descriptive statistics are presented in Tables A1 and A2 of the appendix. The dependent variable *TFR* (Statistics of Sweden) is defined as the sum of the age-specific fertility rate (*SFR*) for women aged 16-49 yrs living in the municipality, as given by Equation (2).

$$TFR = \sum_{i=16}^{49} SFR_i \tag{2}$$

Based on previous results by Macunovich (1998) and Waldorf and Byun (2005), the cohort ratio (*R*) is defined as the number of men aged 35 to 64 yrs divided by the number of men of aged 15 to 34 yrs and is calculated as a 3-year smoothed cohort ratio as shown in Equation (3). The mean age of men at the birth of their first child, which increased from 26.66 in 1970 to 31.46 in 2006, is also examined.

$$R_{jt} = \sum_{\tau=t-2}^{t} M_{35-64,\, j\tau} \;/\; \sum_{\tau=t-2}^{t} M_{15-34,\, j\tau} \tag{3}$$

Whereas *R* measures the relative effect of earnings between generations, the average income responds to the effect of earnings in the short-run dynamics and across municipalities. The average income per capita in each municipality for people aged 20 and above is deflated by CPI (1981=100) and converted to a logarithm because of its log-normality (ln(*I*)). It is assumed to be strongly positively correlated with the average earnings. Income in the data could not be separated by gender; thus, only the mean household effects of earnings on fertility could be measured. The proportion, relative net migration and relative total flow of in-/out-migrated women in the age range 16-49 yrs are considered in the specifications. Preliminary analysis suggested that the relative total flow of in-/out-migrated women is the most highly correlated variable to total fertility rates among the other variables characterizing migration.

To allow for different skills characteristics of local labor markets, the proportion of women with a primary and secondary education of 9 years and less (ISCED97 1) (in the age range 16-49 yrs), the proportion of women with post-graduate education (ISCED97 6) and the proportion of women with post-secondary education of 3 years or more (ISCED97 5A) (aged 20-49 yrs) are included in the model. Thus, boundary skill groups are taken into account in the model.

There are several reasons which can cause the cohort ratio to appear insignificant after including control variables, even though the correlation between them is not high. A larger young generation is obviously connected with greater mobility as the acquisition of education is a key reason for the relocation of young adults. In addition, a higher proportion of the young generation compared to other age groups correlates to a greater proportion of low-skilled people due to a larger number of people aged 16-19, who have not completed their education. However, the proportion of highly skilled people may also be greater among the young generation, particularly, due to increasing accessibility of educational programs over time.

The status of marriage or divorce in fertility was not examined because Hoem et al. (2006) have already published conclusions about the insignificance of these factors on fertility. The proportion of employed women was also not taken into account because of a lack of data for the entire period. However, previous studies have provided a justification for including this information in the analysis (Berinde, 1999; Westerberg, 2006). Following earlier papers, variables such as size of labor force, mean age at first birth and earned income quintiles are expected to be important. These variables are included in some of the specifications.

It is important to eliminate business cycle components from the analysis. Gross domestic product (GDP) in constant prices for Sweden (indexmundi.com) was considered a suitable variable for reflecting business activity changes.

5 Empirical Specification

Space dynamics are taken into account in the model through lagged terms. Residuals are assumed to be not spatially autocorrelated. The total fertility rate dynamics are considered to involve a stationary process with a cyclical component, which required incorporating at least two time lags. Both explanatory variables of interest - income and the cohort ratio - also contain cyclical components. The cyclical component of income is related to business cycles, whereas the cyclical component of the cohort ratio can be explained by demographic cycles. The cohort ratio exhibited a trend associated with aging of the population. Preliminary analysis of the data on income, migration and educational levels showed that they also followed some trends. These findings support the use of a model which incorporates explanatory variables as lags. This is achieved by applying the Arellano-Bond method of estimation for panel data models with a spatially lagged dependent variable. This method is based on the generalized method of moments (GMM), which allows the evaluation of asymptotically efficient estimates under the assumption of autocorrelation between explanatory variables and errors when a stationary process is considered.

The cyclical component of income is modeled using two different approaches. The first approach involved a model with time-specific fixed effects, whereas the second approach employed GDP growth as a control variable for time and business cycles. The development of the spatial first-order and serial second-order panel data model SAR (2,1) for the estimation is presented below in explicit vector form (4). It combines global and local forms of spillovers:

$$\begin{aligned}\mathbf{TFR}_t &= \varphi_1\mathbf{TFR}_{t-1} + \varphi_2\mathbf{TFR}_{t-2} + \gamma\mathbf{W}_{NT}\mathbf{TFR}_{t-1} + \pi\mathbf{R}_{t-1} + \nu\mathbf{V}_{NT}\mathbf{R}_{t-1} + \theta\ln\mathbf{I}_{t-1} \\ &+ \vartheta\mathbf{V}_{NT}\ln\mathbf{I}_{t-1} + \mathbf{X}_{t-1}\beta + \boldsymbol{\varepsilon}_t\end{aligned} \tag{4}$$

where π, θ are parameters associated with the cohort ratio and log-income, respectively, ν and ϑ are time-space autoregressive parameters of the local form of spillovers, φ_k and γ are parameters associated with the time- and space-lagged total fertility rate, respectively, and ε_t is a normally distributed, reciprocally independent vector of errors of size N.

The short-run effect of the lagged *TFR* is described by $\frac{\partial \mathbf{TFR}_t}{\partial \mathbf{TFR}_{t-1}} = \gamma \mathbf{W}' + \varphi \mathbf{I}_N$. The marginal effects of the cohort size ratio and log-income on fertility are $\frac{\partial \mathbf{TFR}_t}{\partial \mathbf{R}_{t-1}} = \nu \mathbf{V}' + \pi \mathbf{I}_N$ and $\frac{\partial \mathbf{TFR}_t}{\partial \ln(\mathbf{I}_{t-1})} = \vartheta \mathbf{V}' + \theta \mathbf{I}_N$, respectively. The first term in these expressions corresponds to spillover or indirect effects, whereas the second term reflects a direct effect of a lagged variable on fertility rate.

The long-run effects presented in Equations (5-7) are derived in Appendix 5. Equations (5-6) give the long-run spatial effects under conditions of a non-zero parameter of the spatially lagged variable. If local spillover parameters ν and ϑ are insignificant Equation (7) can be used to estimate the long-run effects of the cohort ratio and earnings by substituting π and θ in place of β. These formulae imply that even if the short-run spillover effect is absent or the parameter of the spatially weighted variable is equal to zero, the non-zero long-run indirect effect exists in the presence of direct effect. Interpretation of the results and the estimated values of average direct, indirect (spatial) and total effects are given in the next section.

$$\frac{\partial \mathbf{TFR}}{\partial \mathbf{R}} = \left[((1 - \sum_{k=1}^{2} \varphi_k)\mathbf{I}_N - \gamma \mathbf{W})^{-1} (\nu \mathbf{V} + \pi \mathbf{I}_N) \right]' \tag{5}$$

$$\frac{\partial \mathbf{TFR}}{\partial \ln(\mathbf{I})} = \left[((1 - \sum_{k=1}^{2} \varphi_k)\mathbf{I}_N - \gamma \mathbf{W})^{-1} (\vartheta \mathbf{V} + \theta \mathbf{I}_N) \right]' \tag{6}$$

$$\frac{\partial \mathbf{TFR}}{\partial \mathbf{X}} = \left[((1 - \sum_{k=1}^{2} \varphi_k)\mathbf{I}_N - \gamma \mathbf{W})^{-1} \beta \right]' \tag{7}$$

We consider linear coefficients of elasticity $e = b\frac{\bar{x}}{\bar{y}}$ where b to represent the estimate of the parameter for explanatory variable x. The average value of the endogenous variable y alters by e percent when x changes by one percent. Assuming a stationarity condition for *TFR*, the bounds for autoregressive parameters of the model are defined by Equation (8),

$$\left| \frac{\varphi_1 + \gamma\mu_i \pm \sqrt{(\phi_1 + \gamma\mu_i)^2 + 4\varphi_2}}{2} \right| < 1 \tag{8}$$

where $\{\mu_i\}$, i=1,...,N is a set of eigenvalues of the matrix W. The proof of the inequality is shown in Appendix 6. The estimated models are tested for the absence of serial autocorrelation in residuals. Wald values, which are used for accepting or rejecting the hypothesis that each parameter is equal to 0, are shown for each model in Table A4.

6 Results

The results of the estimation for two sets of variables and two types of cyclical component of income are presented in Table A4. The basic specification for the period 1981-2008 included the relative cohort size *R*, log(*I*) and the proportions of in-/out-migrated women aged 16-49 yrs. A more detailed specification for the period 1985-2008 included the same explanatory variables and, in addition, the educational levels of women, i.e., the proportion of women aged 16-49 yrs with more than 3 years post-secondary education and the proportion of women aged 16-49 yrs with less than 9 years education.

The main finding is that the parameter of the weighted lagged *TFR* is significant and positive in both specifications. This suggests a spatial positive autocorrelation of *TFR* applies across municipalities. It means that declining or rising fertility in one municipality affects neighboring municipalities in the same direction. However, the factors generating this spatial autocorrelation could not be extracted separately. Estimated values of the parameters for one specification of the time-specific model, presented in column 8 in Table A4, are shown in Table 1 and discussed in detail below.

For the specification shown in Table 1, the parameter of the spatially lagged *TFR* is greater than the time-lagged variable (0.032 and 0.430, respectively). This result may be wrongly interpreted as showing a larger role of the indirect effect in fertility dynamics than the direct effect. However, the average short-run effect of interactions between municipalities, calculated according to the LeSage and Pace approach (2010), is 0.014, which is less than the time-lagged parameter or the direct effect. Thus, the indirect effect and the direct effect in the short-run explain 0.014 and 0.032 percent of relative changes in total fertility rates, respectively. This means that both reduction and growth of fertility occur in related municipalities on average because of demographic, economic or other reasons. In this case, the long-run effect is absent as the process is considered to be stationary.

Overall, the results weakly support a direct effect according to the inverted Easterlin hypothesis for the analyzed period (1981-2008) because the estimate of the cohort ratio parameter is significant for all the specifications which did not include other labor market characteristics (labor force skills and mobility). However, for the specification considered in Table 1, the parameter for the cohort ratio is not significant. For example, for specification 5 (Table A4), the estimated parameter of the cohort ratio is equal to -0.198, which explains a 0.16% reduction in *TFR* when *R* increases by 1%. The spatially lagged term is not significant in all specifications. However, taking into account the equilibrium condition in the long-run, the direct effect explained 0.19%, indirect effect 0.16%, and total effect 0.35% of the relative change in fertility.

The income parameter is negative in each of the specifications, but the spatially lagged income is found to be insignificant. For the considered specification considered in Table 1, the direct effect of incomes in the short-run is equal to -0.319, whereas it is -0.382 for the long-run. Thus, there appears to be a negative direct effect of earnings on total fertility rates, or equivalently, a dominant substitution effect between wages (labor market) and childbearing (viewed as household production), despite the strong Swedish family policy supporting fertility. This effect manifests as a postponement or acceleration of childbearing in the short-run. The direct short-run effect can be attributed to 0.67% of the variation in fertility. However, consideration of the total long-run effect increased total fertility rates by up to 1.49% (the long-run direct and indirect effects are 0.81 and 0.68%, respectively).

Table 1: Estimates of time-space dynamics in the model describing TFR as a SAR (2,1) process

Variables			The average short-run effect			The average long -run effect		
	Coeff.	St. dev.	Direct	indirect	Total	Direct	Indirect	Total
TFR $_{t-1}$	0.032*	0.015	0.032	0.014	0.046	0	0	0
Weighted TFR $_{t-1}$	0.430***	0.032						
TFR $_{t-2}$	0.086***	0.014	0.086					
$\ln(I_{t-1})$	-0.319*	0.154	-0.319	0	-0.319	-0.382	-0.323	-0.706
Weighted $\ln(I_{t-1})$	0.328	0.272						
R_{t-1}	-0.042	0.065						
Weighted R_{t-1}	-0.708	0.610						
(The relative total flow of out-/in-migrated women aged 16-49)$_{t-1}$	1.328**	0.526	1.328			1.594	1.343	2.937
(The proportion of women aged 16-49 yrs with more than 3 yrs post-secondary)$_{t-1}$	0.978***	0.281	0.978			1.174	0.989	2.163
(The proportion of women aged 16-49 yrs with less than 9 yrs education)$_{t-1}$	1.067***	0.274	1.067			1.281	1.079	2.360
Intercept	1.798***	0.541	1.798					
Number of observations				6322				
Wald χ^2				9615				
Stationarity				*Accepted*				

Note: Asterisks denote *p*-values of less than 0.0001 (***), 0.01 (**) or 0.05 (*).

Among the control variables tested, the most important is the sum of in- and out-migrated people in the population, i.e., the relative total flow. A high degree of labor mobility in both directions may reflect several processes, such as changes in age-gender structure, levels of education in a municipality, the number of matches resulting in cohabitation and marriage, and increasing demand and supply values in the labor market over time. The main factor cannot be extracted, but evidently, internal migration is positively related to childbearing, giving significant direct, indirect and total effects in the long-run 1.594, 1.343 and 2.937, respectively. Since the numbers of in- and out-migrated people compared to the size of the population is rather low, the total long-run effect only captured 0.001% of the relative changes in fertility.

Municipalities with a large proportion of women with educational levels higher or lower than the mean level exhibited the highest total fertility rates. This result agrees with previous papers (Anderson et al., 2003; Berinde, 1999) and probably reflects the low opportunity cost of childbearing that women with low educational levels experience. The positive estimate for the contribution of highly educated women can be attributed to the higher benefits they receive during maternity leave and greater opportunities for flexibility in distributing their time between employment and taking care of a child. In comparison, the long-run effects of the proportions of high and low educated women on total fertility rates are higher by almost two fold and totally explained 0.105% and 0.054% of the long-run variation in fertility, respectively.

Substituting estimates of the autoregressive parameters and boundary eigenvalues for **W** in Equation (8) proves that the process, described in the model, is stationary. In contrast to a previous study for Italy (Waldorf and Franklin, 2002), where total fertility rates are linked to changes in the average age of first births, no such relationship is found for Sweden.

7 Conclusions

There is no doubt that changes in fertility can be explained by economic factors. A large number of papers have provided theoretical and empirical explanations of reproductive behavior, considering female employment and alternative investment choices, such as gaining an education.

The present paper makes an important contribution by reporting an analysis of the relationship between changes in childbearing and several factors that are important for settlements located near to each other. The space autoregressive component was found to be positively significant in all of the specifications employed. This suggests that, for a given municipality, the whole set of factors which can affect total fertility rates, cause it to change in the same direction as in surrounding municipalities.

The main aim of this paper was to empirically test the relationship between fertility and earnings in time and space dimensions. The findings support the inverted Easterlin hypothesis. The results revealed a short-run direct effect, and long-run direct and indirect effects. In the short-run earnings negatively affect fertility within municipalities, but an indirect effect across space was not found. The specification of the model allows the long-run spatial effect to be evaluated in addition to the direct effect. The results indicated that the long-run spatial effect was negative, which can be explained by a dominant substitution effect governing the choice between female labor supply and childbearing as a household's production, despite the fact that family policy in Sweden provides good opportunities to combine them both.

A stationarity condition was adopted in the present study in the form of an inequality, which defined bounds for the autoregressive parameters by considering the unit root of the weight matrix for the spatially-autoregressive parameter. Using these conditions, the stationarity of fertility in time and space dynamics, measured by total fertility rates, was empirically proved.

The observed long-run effects of the explanatory variables in the SAR (2,1) model derived in this paper imply that direct, indirect and total effects should be considered. It was empirically shown that the spatial effect in the long-run almost doubles the influences of the log-income and cohort ratio on fertility compared to the short-run. Thus, the results clearly show that spatial disturbances are important in modeling fertility processes.

References

Andersson, G. (1999) *Childbearing Trends in Sweden 1961–1997*. European Journal of Population 15: 1–24.

Andersson, G., Duvander, A., Hank, K. (2003) *Do Child Care Characteristics Influence Continued Childbearing in Sweden? An Investigation of the Quantity, Quality, and Price Dimension*. MPIDR WORKING PAPER WP 2003-013 MAY 2003.

Anselin L. (2002) *Under the hood. Issues in the Specification and Interpretation of Spatial Regression Models.* Agricultural Economics 27: 247–267.

Artzrouni M.A., Easterlin R.A. (1982) *Birth History, Age Structure, and Post World War II Fertility in Ten Developed Countries: An Exploratory Empirical Analysis*. Genus 38 (3–4): 81–99.

Baird, A.J. (1987) *A Note on the Easterlin Model of Fertility in Northwestern Europe and the United States: 1950–1981*. International Journal of Comparative Sociology 23 (1–2): 57–68.

Becker, G.S. (1960) *An Economic Analysis of Fertility - Demographic and Economic Change in Developed Countries.* Universities-National Bureau: 225–256.

Becker, G.S. (1992) *Fertility and the Economy*. Journal of Population Economics 5 (3): 185–201.

Becker, G.S., Barro, R.J. (1988) *A Reformulation of the Economic Theory of Fertility*. The Quarterly Journal of Economics 103 (1): 1–25.

Berinde, D. (1999) *Pathways to a Third Child in Sweden.* European Journal of Population 15: 349–378.

Bongaarts, J., Watkins, C.S. (1996) *Social Interactions and Contemporary Fertility Transitions.* Population and Development Review 22 (4): 639–682.

Brueckner, J.K. (2003) *Strategic Interaction among Governments: An Overview of Empirical Studies*. International Regional Science Review 26 (2): 175–188.

de Castro, M.C. (2007) *Spatial Demography: An Opportunity to Improve Policy Making at Diverse Decision Levels*. Population Resource Policy Review 26: 477–509.

Easterlin, R.A. (1966) *On the Relation of Economic Factors to Recent and Projected Fertility Changes*. Demography 3: 135–153.

Elhorst, J.P. (2001) *Dynamic Models in Space and Time*. Geographical Analysis 33 (2): 119–140.

Eliasson, K., Westerlund, O., Åström, J. (2007) *Flyttning och pendling i Sverige*. Fritzes, Stockholm. 219 p.

Hoem J. M., Neyer, G., Andersson, G. (2006) *Educational Attainment and Ultimate Fertility among Swedish Women Born in 1955-59*. MPIDR WORKING PAPER WP 2006-004. MARCH 2006.

Hoem, B. (2000) *Entry into Motherhood in Sweden: the Influence of Economic Factors on the Rise and Fall in Fertility, 1986-1997*. Demographic Research, vol. 2, art. 4.

LeSage, J.P., Pace, R.K. (2010) *Spatial Econometric Models* in M.M. Fisher and A. Getis (eds.), Handbook of Applied Spatial Analysis: Software, Tools, Methods and Applications. 355-376.

Macunovich, D.J. (1998) *Fertility and the Easterlin hypothesis: An Assessment of the Literature*. Population Economy 11: 53–111.

McNicoll, G., (1980) *Institutional Determinants of Fertility Change.* Population and Development Review 6 (3): 441–462.

Pampel, F.C. (1993) *Relative Cohort Size and Fertility: The Socio-Political Context of the Easterlin Effect*. American Sociological Review 58 (August): 496–514.

Sevilla J.P. (2007) *Fertility and Relative Cohort Size*. Arbetsrapport. Institutet för Framtidsstudier, 2007:11.

Silvester, J.R. (2000) *Determinants of Block Matrices*. The Mathematical Gazette. 84 (501): 460–467.

Waldorf B., Byun P. (2005) *Meta-analysis of the Impact of Age Structure on Fertility*. Journal of Population Economy 18: 15–40.

Waldorf, B., Franklin R. (2002) *Spatial dimension of the Easterlin hypothesis: fertility variations in Italy*. Journal of Regional Science 42 (3): 549–578.

Weeks, J.R., Getis, A., Hill, A.G., Gadalla, M.S., Rashed, T. (2004) The *Fertility Transition in Egypt: Intraurban Patterns in Cairo*. Annals of the Association of American Geographers 94 (1): 74–93.

Westerberg, T. (2006) *More Work, Less Kids – The Relationship Between Market Experience and Number of Children.* Umeå Economic Studies. No. 683.

Woods, R. (1984) *Social Class Variations in the Decline of Marital Fertility in Late Nineteenth-Century London.* Geografiska Annaler. Series B, Human Geography 66 (1): 29–38.

Wright, R.E. (1989) *The Easterlin Hypothesis and European Fertility Rates.* Population and Development Review 15 (1): 107–122.

Yua, J., de Jong, R., Lee, L. (2008) *Quasi-maximum Likelihood Estimators for Spatial Dynamic Panel Data with Fixed Effects when both n and T are Large*. Journal of Econometrics 146: 118–134.

Appendix

Table A1: List of variables

Total fertility rate (TFR)	**Total fertility rate by region, gender and period: The data are based on the Historical Population and Multi-Generation registers. The Multi-Generation Register is updated continuously with links between mother/child and father/child. The statistics could differ slightly from official statistics depending on yearly corrections to the registers. Thus, the whole series is updated each year for the entire period.**
Relative cohort size (*R*)	The number of men aged 35 to 64 yrs divided by the number of men of aged 15 to 34 yrs smoothed over 3 years.
The proportion of women aged 16-49 yrs with a given level of education • **less than 9 years (ISCED97 1) of education** • **post-graduate education (ISCED97 6)** • **post-secondary education of more than 3 years (ISCED97 5A)**	Women aged 16-49 yrs with a given level of education / total number of women aged 16-49. Level of educational attainment by region, gender, age, income class and time: The classification by level of educational attainment follows the Swedish National Educational classification (SUN)
Income	Total and average income for residents in Sweden on 31/12 in thousands by region, gender, age, income class and time
Income_CPI	Income adjusted by CPI to the basic level in 1981
Numbers of in- and out-migrated women	Numbers of in- and out-migrated women by region, age and time. When calculating age-specific rates per 1 000 of the mean population, the mean population for the year of birth should be used. Age refers to the age attained by the end of the year, i.e., in principle, the number of years since birth.
The relative balance of in-/out-migrated women aged 16-49 yrs	(In-migrated women aged 16-49 yrs – out-migrated women aged 16-49 yrs)/Total number of women aged 16-49 yrs
The relative flow of in-/out-migrated women aged 16-49 yrs	(In-migrated women in aged 16-49 yrs + out-migrated women aged 16-49 yrs)/Total number of women aged 16-49 yrs
Mean age of woman at birth of the first child (AAFB)	Mean age at birth of the first child by region and time: The data were based on the Historical population register and the Multi-Generation Register. The Multi-Generation Register is updated continuously with links between mother/child and father/child. The link between father and child is registered with a time delay compared to the link between mother and child. Consequently, the numbers are uncertain for fathers in 2007. The statistics could differ slightly from official statistics depending on yearly corrections to the registers. Thus, the whole series is updated each year for the entire period.
Population size (100 thou.)	Mean population (by year of birth) by region, age, sex and time: When summing up the mean population (e.g., in 10-year groups), any rounded figures are accumulated. This may cause the accumulated figure to be somewhat higher than the totals. The mean population refers to the mean value of, for instance, the number of 5-year olds at the end of year *n* and the number of 6-year olds at the end of year *n*+1.
Gross domestic product (GDP) in constant prices	Annual percentages of constant price GDP are year-on-year changes; the base year is 1990.

Table A2: Descriptive statistics for 276 municipalities during 1970-2008

Variable	Obs	Mean	Std. Dev.	Min	Max
Total fertility rate (TFR)	7452	1.929	0.301	0.91	3.31
Mean age at birth of the first child, female parents (AAFB)	7452	26.596	1.525	22.33	33.07
Mean age at birth of the first child, male parents (AAFB)	40	29.113	1.566	26.66	31.46
Income to CPI, 1981=100%	4968	58.189	9.427	39.770	180.705
The relative net migration of in-/out-migrated women aged 16-49 yrs	7422	-0.001	0.006	-0.022	0.042
The relative total flow of in-/out-migrated women aged 16-49 yrs	7422	0.061	0.019	0.023	0.205
Relative cohort (*R*)	7452	1.572	0.249	0.952	2.369
The proportion of women aged 16-49 yrs with less than 9 years education	7452	0.044	0.046	0.101	0.231
The proportion of women with post-graduate education	7452	0.002	0.003	0.004	0.047
The proportion of women aged 16-49 yrs with more than 3 years post-secondary education	7452	0.094	0.064	0.030	0.456
Population size (100 thousands)	7452	0.312	0.566	0.028	8.026
GDP in constant prices[1]	28	2.375	1.825	-2.058	4.66

[1] data for the period 1981-2008

Appendix A3. The municipality restructuring in 1992

In 1992, municipality reform was performed in Sweden, which resulted in the number and boundaries of municipalities being changed. The municipalities which were established or annulled as a consequence of this reform were excluded from the present analysis. Thus, the total number of municipalities included was 276. Weight matrices based on spherical distances were unchanged by the reform because the municipal center had the same geographical coordinates. Weight matrices based on contiguity were changed, but the change in boundaries was not so large as to give misleading results. After the reform, the individual municipal populations changed by more than 2% in only 9 municipalities (only in Vaxholm and Värmdö did the population change by more than 3%), but the total change in population was not more than 5%. The variables used were merged by Statistics Sweden (SCB).

Table A4: GMM estimates for *TFR*. The model with time-specific fixed effects for the period 1981-2008.

Variables	No spatial effects				SAR				SAR with exogenous spatial interaction effect							
	1		2		3		4		5		6		7		8	
	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.	Coeff.	St. dev.
TFR $_{t-1}$	0.095***	0.015	0.044**	0.016	0.033*	0.015	0.033*	0.015	0.033*	0.015	0.037*	0.015	0.044**	0.016	0.032*	0.015
TFR $_{t-2}$	0.148***	0.014	0.100***	0.015	0.084***	0.014	0.086***	0.014	0.084***	0.014	0.092***	0.015	0.100***	0.015	0.086***	0.014
***Weighted* TFR $_{t-1}$**					0.422***	0.032	0.431***	0.032	0.423***	0.032					0.430***	0.032
ln(I_{t-1})	0.961***	0.100	0.120	0.130	-0.708	0.194	-0.220*	0.130	-0.759***	0.200	-0.848***	0.203	-0.017	0.156	-0.319*	0.154
***Weighted* ln(I_{t-1})**									0.306	0.269	0.247	0.274	0.398	0.279	0.328	0.272
R_{t-1}	-0.225***	0.063	-0.014	0.066	-0.215*	0.059	-0.050	0.065	-0.198***	0.061	-0.251***	0.062	-0.011	0.067	-0.042	0.065
Weighted R_{t-1}									-0.660	0.597	-0.458	0.608	-0.639	0.625	-0.708	0.610
(The relative flow of in/out-migrated women aged 16-49 yrs)$_{t-1}$	2.398***	0.550	1.889***	0.537	1.435**	0.518	1.320**	0.526	1.444**	0.518	1.902***	0.528	1.893***	0.537	1.328**	0.526
(The proportion of women aged 16-49 yrs with more than 3 years of post-secondary education) $_{t-1}$			1.677***	0.265			1.078***	0.263					1.500***	0.284	0.978***	0.281
(The proportion of women aged 16-49 yrs with less than 9 years of education)$_{t-1}$			1.498***	0.272			1.095***	0.268					1.428***	0.279	1.067***	0.274
Intercept	-2.076***	0.349	1.105*	0.472	4.126***	0.828	1.740***	0.463	4.185***	0.991	5.299***	1.005	0.946*	0.551	1.798***	0.541
Number of observations	6596		6322		6596		6322		6596		6596		6322		6322	
Wald χ^2	8406		9007		9963		9612		9936		9388		9007		9615	

Note: Asterisks indicate *p*-values less than 0.0001 (***), 0.01 (**), and 0.05 (*).

Appendix 5

The long-run effect of an explanatory variable in the SAR (2,1) model

Consider the model SAR (2,1) with a second-order autoregressive disturbance, a first-order spatial disturbance weighted by a spatial weights matrix **W** of size $N\times N$ and a first-order lagged common factor weighted by some spatial matrix **V** of size $N\times N$.

$$\mathbf{y}_t = \varphi_1 \mathbf{y}_{t-1} + \varphi_2 \mathbf{y}_{t-2} + \gamma \mathbf{W}\mathbf{y}_{t-1} + \eta \mathbf{x}_{t-1} + \delta \mathbf{V}\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t \,, \tag{A5:1}$$

where ε_t is a "white noise" vector of size $N\times 1$, $\mathbf{y}_t$ is a vector of size $N\times 1$ of observations of an endogenous variable for every spatial unit at time t, x_t is a vector of explanatory variables at time t, and φ, γ, δ and η are parameters. If $\mathbf{y}_t$ converges to equilibrium, the equilibrium condition is given by the following equation:

$$((1-\varphi_1-\varphi_2)\mathbf{I} - \gamma\mathbf{W})\mathbf{y} = (\eta\mathbf{I} + \delta\mathbf{V})\mathbf{x} \tag{A5:2}$$

where **I** is the identity matrix of size $N\times N$.

Thus, the long-run impact of x on y yields

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left[((1-\varphi_1-\varphi_2)\mathbf{I} - \gamma\mathbf{W})^{-1}(\eta\mathbf{I} + \delta\mathbf{V}) \right]^{\prime} \mathbf{y} \tag{A5:3}$$

Appendix 6

Stationarity condition in the SAR(2,1) model

Consider the following stationarity condition in the SAR(2,1) model described in Appendix 5:

$$\mathbf{y}_t = \varphi_1 \mathbf{y}_{t-1} + \varphi_2 \mathbf{y}_{t-2} + \gamma \mathbf{W}\mathbf{y}_{t-1} + \eta \mathbf{x}_{t-1} + \delta \mathbf{V}\mathbf{x}_{t-1} + \boldsymbol{\varepsilon}_t \qquad \text{(A6:1)}$$

The vector auto-regression form (VAR) for this model yields:

$$\begin{pmatrix} \mathbf{y}_t \\ \mathbf{y}_{t-1} \end{pmatrix} = \begin{pmatrix} \varphi_1 \mathbf{I} + \gamma \mathbf{W} & \varphi_2 \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{y}_{t-1} \\ \mathbf{y}_{t-2} \end{pmatrix} + \begin{pmatrix} (\eta \mathbf{i} + \delta \mathbf{V})\mathbf{x}_{t-1} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{0} \end{pmatrix} \qquad \text{(A6:2)}$$

where **i** is a vector of units. Let matrix **A** be defined as

$$\mathbf{A} = \begin{pmatrix} \varphi_1 \mathbf{I} + \gamma \mathbf{W} & \varphi_2 \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \qquad \text{(A6:3)}$$

The stationarity condition implies $|\lambda_i|<1$ when λ_i (i=1,...,2n) is a characteristic root of the matrix A, or a root of the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$, which is a polynomial in λ.

The matrix $\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} \varphi_1 \mathbf{I} + \gamma \mathbf{W} - \lambda \mathbf{I} & \varphi_2 \mathbf{I} \\ \mathbf{I} & -\lambda \mathbf{I} \end{pmatrix}$ has a property of block matrix, described by Silvester (2000, p. 463). Accordingly,

$$\left|\mathbf{A}-\lambda\mathbf{I}\right| = \begin{vmatrix} \varphi_1\mathbf{I}+\gamma\mathbf{W}-\lambda\mathbf{I} & \varphi_2\mathbf{I} \\ \mathbf{I} & -\lambda\mathbf{I} \end{vmatrix} = \left|(\varphi_1\mathbf{I}+\gamma\mathbf{W}-\lambda\mathbf{I})(-\lambda\mathbf{I})-\varphi_2\mathbf{I}\right| = \left|\lambda^2\mathbf{I}-(\varphi_1\mathbf{I}+\gamma\mathbf{W})\lambda-\varphi_2\mathbf{I}\right| \qquad \text{(A6:4)}$$

Let **M** be a matrix containing the characteristic roots $\{\mu_i\}$ (i=1,...,N) of the weight matrix **W** of size $N\times N$. According to the basic properties of a block matrix, |**M**|=|**W**|, and there exists a nonsingular matrix **R** of characteristic vectors of size $n\times n$ such that $\mathbf{RMR}'=\mathbf{W}$, $\mathbf{RR}'=\mathbf{I}$. Using this decomposition and the given properties, the characteristic equation for **A** can be transformed as follows:

$$\left|\lambda^2\mathbf{I}-(\varphi_1\mathbf{I}+\gamma\mathbf{W})\lambda-\varphi_2\mathbf{I}\right| = \left|\lambda^2\mathbf{RIR}^{-1}-(\varphi_1\mathbf{RIR}^{-1}+\gamma\mathbf{RMR}^{-1})\lambda-\varphi_2\mathbf{RIR}^{-1}\right| = \left|\mathbf{R}\left[\lambda^2\mathbf{I}-(\varphi_1\mathbf{I}+\gamma\mathbf{M})\lambda-\varphi_2\mathbf{I}\right]\mathbf{R}^{-1}\right| = 0 \qquad \text{(A6:5)}$$

Since **R** is a nonsingular matrix and $\lambda^2\mathbf{I}-(\varphi_1\mathbf{I}+\gamma\mathbf{M})\lambda-\varphi_2\mathbf{I}$ is a diagonal matrix, the determinant is equal to zero and λ is a root of the polynomial $\lambda^2-(\varphi_1+\gamma\mu_i)\lambda-\varphi_2$.

Thus,

$$\lambda_j = \frac{\varphi_1+\gamma\mu_i \pm \sqrt{(\varphi_1+\gamma\mu_i)^2+4\varphi_2}}{2}, \ j=1,...,2n \qquad \text{(A6:6)}$$

Finally, the stationarity condition can be written as Equation (8).